Moduli of vector bundles over real algebraic curves

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Vector bundles on algebraic curves

What is the genus of this surface?

- Let X be a compact, connected **Riemann surface** of genus $g \geqslant 0$.
- For all integers $r \geqslant 1$ and $d \in \mathbb{Z}$, there is a moduli variety $\mathcal{M}_X(r, d)$ of **semistable holomorphic vector bundles** of rank r and degree d over X. When $r = 1$, one has $\mathcal{M}(1, d) = \mathrm{Pic}_d(X)$, which is a complex torus $\mathbb{C}^g/\mathbb{Z}^{2g}$.
- In joint work with Melissa Liu (Columbia University), we have been interested in the topology of $\mathbb{R}\mathcal{M}_X(r, d)$ when X is equipped with a **real structure** $\sigma : X \to X$.

Outline of the talk

1. Moduli spaces of holomorphic vector bundles
2. The recursive formula of Atiyah and Bott
3. Moduli spaces of real vector bundles
4. The recursive formula in the real case

Topological invariants

- If $\mathscr{L}$ is a holomorphic line bundle on X, its degree $d \in \mathbb{Z}$ can be defined as
$$\deg(\mathscr{L}) := \text{Zeros}(s) - \text{Poles}(s)$$
where s is any meromorphic section of $\mathscr{L}$. If $\mathcal{E}$ is a holomorphic line bundle, then
$$\deg(\mathcal{E}) := \deg(\det(\mathcal{E})).$$
- The **rank and degree** of a holomorphic vector bundle $\mathcal{E}$ are complete topological invariants.
- In what follows, we fix a C^∞ complex vector bundle E (of rank r and degree d) and we look at the space of holomorphic structures $\mathscr{C}$ on E.
- If E is equipped with a C^∞ **Hermitian metric** h, then $\mathscr{C}$ can be identified with the space of **unitary connections** for that metric.

Yang-Mills connections

- Two unitary connections on E define isomorphic holomorphic structures on E if and only if they lie in a same $\mathscr{G}_E$-orbit, where $\mathscr{G}_E$ is the **complex gauge group** of E (the group of all complex automorphisms of E).
- By a theorem due to Donaldson (1984) and Daskalopoulos-Råde (1988), a $\mathscr{G}_E$-orbit of unitary connections $\mathcal{O}$ defines a semistable holomorphic structure on E if and only if its *closure* $\overline{\mathcal{O}}$ contains a **minimal Yang-Mills connection**.

$$F_A = \begin{pmatrix} \frac{d}{r} & & \\ & \ddots & \\ & & \frac{d}{r} \end{pmatrix} \in \mathrm{Herm}(E, h)$$

- Yang-Mills connections are by definition the critical points of the Yang-Mills functional $A \mapsto \int_X \|F_A\|^2$, defined in the set of all unitary connections $\mathscr{C}$.

The GIT picture

- There is an invariant-theoretic interpretation of the above characterization of semistable orbits. It rests on the additional observation that every **orbit closure** contains exactly one closed orbit.
- Together with the fact that an orbit is closed if and only if it contains a Yang-Mills connection, this implies that the relation

$$\mathcal{O}_1 \sim \mathcal{O}_2 \text{ if } \overline{\mathcal{O}_1} \cap \overline{\mathcal{O}_2} \neq \emptyset$$

  is an **equivalence relation**.
- The moduli space $\mathcal{M}_X(r, d)$ is the space $\mathscr{C}_{ss} /\!/ \mathscr{G}_E$ of so-called *S*-**equivalence classes of semistable orbits**. Equivalently, it is the space of closed semistable orbits.

Slope stability

- The GIT picture is useful to get a sense of why a moduli space should exist, but less so to give intuition on (semi)stable objects.
- Mumford's definition of (semi)stability is

$$\forall\, \mathcal{F} \subset \mathcal{E},\ (\mathcal{F} \neq 0 \wedge \mathcal{F} \neq \mathcal{E}) \Rightarrow \frac{\deg(\mathcal{F})}{\mathrm{rk}(\mathcal{F})}(\leqslant)\frac{\deg(\mathcal{E})}{\mathrm{rk}(\mathcal{E})} = \frac{d}{r}\,.$$

- When $r \wedge d = 1$, every semistable vector bundle is in fact stable, with automorphism group $\mathrm{Aut}(\mathcal{E}) \simeq \mathbb{C}^*$. Equivalently, every semistable orbit is closed, with isotropy group $\mathcal{Z}(\mathscr{G}_E) \simeq \mathbb{C}^*$. In this case, the moduli space is an **orbit space**:

$$\mathcal{M}_X(r, d) = \mathscr{C}_s/\mathscr{G}_E\ .$$

Topology of the moduli space

- Thanks to Atiyah and Bott (1983), we have a **recursive formula** to compute the Betti numbers of $\mathcal{M}_X(r,d)$ when $r \wedge d = 1$.
- More precisely, we can compute:
  - The Poincaré series of the **moduli stack** $[\mathscr{C}/\mathscr{G}_E]$ of all holomorphic vector bundles of rank r and degree d.
  - The Poincaré series of the **moduli stack** $[\mathscr{C}_{ss}/\mathscr{G}_E]$ of semistable holomorphic vector bundles of rank r and degree d.
  - When $r \wedge d = 1$, the Poincaré polynomial of the **moduli variety**

$$\mathcal{M}_X(r,d) = \mathscr{C}_s/\mathscr{G}_E.$$

- In joint work with Melissa Liu, we perform analogous computations for moduli spaces of **real and quaternionic vector bundles**.

Outline of the talk

Equivariant cohomology

- Since $[\mathscr{C}/\mathscr{G}_E]$ and $[\mathscr{C}_{ss}/\mathscr{G}_E]$ are quotient stacks, their Poincaré series is computed equivariantly: Atiyah and Bott have computed $P_t^{\mathscr{G}_E}(\mathscr{C})$ and $P_t^{\mathscr{G}_E}(\mathscr{C}_{ss})$, both with **rational** and **integer coefficients**.
- Since $\mathscr{C}$ is an affine space (of unitary connections),

$$P_t^{\mathscr{G}_E}(\mathscr{C}) = P_t(B\mathscr{G}_E)$$

  where $B\mathscr{G}_E$ is the **classifying space** of the gauge group.
- The action of $\mathscr{G}_E$ on $\mathscr{C}_s$ is not free (since the center $\mathcal{Z}(\mathscr{G}_E) \simeq \mathbb{C}^*$ acts trivially), but the action of $\overline{\mathscr{G}_E} := \mathscr{G}_E/\mathbb{C}^*$ is **free**, so

$$P_t^{\overline{\mathscr{G}_E}}(\mathscr{C}_s) = P_t(\mathscr{C}_s/\mathscr{G}_E).$$

The coprime case

- When $r \wedge d = 1$, $\mathscr{C}_s = \mathscr{C}_{ss}$ and $H^*(B\mathscr{G}_E; \mathbb{Q}) \to H^*(B\mathbb{C}^*; \mathbb{Q})$ is **surjective**.
- As a consequence, the fibration

$$B\mathbb{C}^* \to [\mathscr{C}_{ss}/\mathscr{G}_E] \to \mathscr{C}_{ss}/\overline{\mathscr{G}_E} = \mathcal{M}_X(r, d)$$

is cohomologically trivial in this case, so

$$P_t(\mathcal{M}_X(r, d)) = (1 - t^2) P_t^{\mathscr{G}_E}(\mathscr{C}_{ss}).$$

- The point is that it now suffices to compute $P_t^{\mathscr{G}_E}(\mathscr{C}_{ss})$. This is where the recursive approach will be used (without, in fact, assuming $r \wedge d = 1$).

Stratification of the space of all bundles

- The affine space $\mathscr{C}$ is stratified according to the **Harder-Narasimhan type** of holomorphic vector bundles of rank r and degree d:

$$\mathscr{C} = \bigsqcup_{\mu \in I_{r,d}} \mathscr{C}_\mu$$

- The set $I_{r,d}$ consists of all tuples

$$\mu = (\underbrace{\mu_1, \ \ldots \ , \mu_1}_{r_1 \text{ times}}, \ \ldots \ , \underbrace{\mu_\ell, \ \ldots \ , \mu_\ell}_{r_\ell \text{ times}})$$

  such that $\ell \geqslant 1$, $r_1 + \cdots + r_\ell = r$ and $d_1 + \cdots + d_\ell = d$.

- The HN type of a bundle $\mathcal{E} \in \mathscr{C}$ is defined by its **Harder-Narasimhan filtration**.

Harder-Narasimhan filtration

- By a theorem due to Harder and Narasimhan (1975), for all $\mathcal{E} \in \mathscr{C}$, there exists a **unique filtration**

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_\ell = \mathcal{E}$$

  such that:

  1. For all $i \geqslant 1$, the vector bundle $\mathcal{E}_i/\mathcal{E}_{i-1}$ is semistable.
  2. The slopes $\mu_i := \mu(\mathcal{E}_i/\mathcal{E}_{i-1})$ satisfy

$$\mu_1 > \cdots > \mu_\ell$$

- The Harder-Narasimhhan type $\mu := (\mu_1, \ldots, \mu_1, \ldots, \mu_\ell, \ldots, \mu_\ell)$ therefore corresponds to the family $(r_i, d_i)_{1 \leqslant i \leqslant \ell}$ of **topological invariants** of the successive quotients $\mathcal{E}_i/\mathcal{E}_{i-1}$.

An equivariantly perfect stratification

- The action of the gauge group $\mathscr{G}_E$ preserves each stratum $\mathscr{C}_\mu$. Moreover, each stratum has **finite codimension**

$$d_\mu = \sum_{1 \leqslant i < j \leqslant \ell} r_i r_j (\mu_i - \mu_j) + (g - 1)).$$

- By results of Atiyah-Bott (and Desale-Ramanan in 1979), the stratification is **equivariantly perfect** over the rationals:

$$P_t^{\mathscr{G}_E}(\mathscr{C}) = \sum_{\mu \in I_{r,d}} t^{2d_\mu} P_t^{\mathscr{G}_E}(\mathscr{C}_\mu).$$

- Equivalently:

$$P_t^{\mathscr{G}_E}(\mathscr{C}_{ss}) \quad = \quad P_t(B\mathscr{G}_E) \quad - \sum_{\mu \in I_{r,d} \ \mid \ d_\mu > 0} t^{2d_\mu} P_t^{\mathscr{G}_E}(\mathscr{C}_\mu).$$

The recursive step

- By definition of the HN filtration, we have, for all HN type μ, a morphism of stacks
$$\begin{array}{ccc} \mathscr{C}_\mu & \to & \mathscr{C}_{ss}(r_1, d_1) \times \cdots \times \mathscr{C}_{ss}(r_\ell, d_\ell) \\ \mathcal{E} & \mapsto & (\mathcal{E}_1/\mathcal{E}_0, \ldots, \mathcal{E}_\ell/\mathcal{E}_{\ell-1}) \end{array}$$
- Atiyah and Bott have shown this morphism has **contractible fibres**.
- Since $\mathcal{E} \in \mathscr{C}_\mu$ admits a reduction of structure group of a standard parabolic subgroup of $\mathbf{GL}(r, \mathbb{C})$, that morphism induces an isomorphism of $\mathbb{Q}$-vector spaces
$$H^*_{\mathscr{G}_E}(\mathscr{C}_\mu; \mathbb{Q}) \simeq H^*_{\mathscr{G}(r_1,d_1)}(\mathscr{C}_{ss}(r_1, d_1); \mathbb{Q}) \otimes \cdots \otimes H^*_{\mathscr{G}(r_\ell,d_\ell)}(\mathscr{C}_{ss}(r_\ell, d_\ell); \mathbb{Q}).$$
- Hence an equality $P_t^{\mathscr{G}_E}(\mathscr{C}_\mu) = \prod_{i=1}^{\ell} P_t^{\mathscr{G}(r_i,d_i)}(\mathscr{C}_{ss}(r_i, d_i))$.

Cohomology of the classifying space

- We now have the formula

$$P_t^{\mathscr{G}_E}(\mathscr{C}_{ss}) \;=\; P_t(B\mathscr{G}_E) \;-\; \sum_{\mu\in I_{r,d}\;|\;d_\mu>0} t^{2d_\mu}\prod_{i=1}^{\ell} P_t^{\mathscr{G}(r_i,d_i)}(\mathscr{C}_{ss}(r_i,d_i)).$$

- Since $r_1+\cdots+r_\ell=r$, we will eventually reach the case when $r_i=1$, which is essentially that of a Picard variety, a complex torus of dimension g hence with Poincaré polynomial $(1+t)^{2g}$.
- There remains to compute $P_t(B\mathscr{G}_E)$, which represents the Poincaré series of the stack of all holomorphic vector bundles of rank r and degree d.

Cellular decomposition

- Recall that (E, h) is a fixed C^∞ complex vector bundle of rank r and degree d and equipped with a Hermitian metric.
- Atiyah and Bott have observed that there is a homotopy equivalence

$$B\mathscr{G}_E \sim \mathrm{Map}_d(X; B\mathbf{U}(r))$$

  where $\mathrm{Map}_d(X; B\mathbf{U}(r))$ is the connected component of $\mathrm{Map}(X; B\mathbf{U}(r))$ containing a map $f : M \to B\mathbf{U}(r)$ such that $E \leftrightarrow f^* E\mathbf{U}(r)$.
- Then, using a standard cellular decomposition of X,

$$\bigvee_{i=1}^{2g} S^1 \hookrightarrow X \to S^2$$

  one can define two fibrations.

Two fibrations

- The first fibration comes from the evaluation map at the 0-cell x_0:

$$\mathrm{Map}_d^\bullet(X, B\mathbf{U}(r)) \to \mathrm{Map}_d(X, B\mathbf{U}(r)) \to B\mathbf{U}(r).$$

- The second fibration comes from applying the pointed mapping functor to the cofibration defined by the cellular decomposition:

$$\mathrm{Map}_d^\bullet(S^2, B\mathbf{U}(r)) \to \mathrm{Map}_d^\bullet(X, B\mathbf{U}(r)) \to \mathrm{Map}^\bullet(\vee_{i=1}^{2g} S^1, B\mathbf{U}(r)).$$

- By an application of the Leray-Hirsch theorem, these two fibrations are cohomologically trivial over $\mathbb{Q}$.

Poincaré series

- Note that

$$\mathrm{Map}^\bullet(\vee_{i=1}^{2g} S^1, B\mathbf{U}(r)) \sim \prod_{i=1}^{2g} \Omega B\mathbf{U}(r) \sim \mathbf{U}(r)^{2g}$$

  and that

$$\mathrm{Map}_d^\bullet(S^2, B\mathbf{U}(r)) \sim \Omega^2 B\mathbf{U}(r)_d \sim \Omega \mathbf{SU}(r).$$

- So Atiyah-Bott's result says that

$$\begin{aligned} P_t(B\mathscr{G}_E) &= P_t(\Omega\mathbf{SU}(r))P_t(\mathbf{U}(r))^{2g}P_t(B\mathbf{U}(r)) \\ &= \frac{\prod_{j=1}^{r}(1+t^{2j-1})^{2g}}{\prod_{j=1}^{r-1}(1-t^{2j})\prod_{j=1}^{r}(1-t^{2j})} \end{aligned}$$

- In fact, their result is more precise and gives generators of $H^*(B\mathscr{G}_E;\mathbb{Q})$, which is how the Leray-Hirsch theorem is applied.

Summary

$$\begin{array}{ccccc} \Omega\mathbf{SU}(r) & \longrightarrow & B\mathscr{G}_E(x_0) & \longrightarrow & \mathbf{U}(r)^{2g} \\ & & \big\downarrow & & \\ & & B\mathscr{G}_E & & \\ & & \big\downarrow \scriptstyle{\mathrm{ev}_{x_0}} & & \\ & & B\mathbf{U}(r) & & \end{array}$$

Can we do something similar over $\mathbb{R}$?

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Real structures

- Let us consider a real structure $\sigma : X \to X$, meaning an anti-holomorphic involution on X. We refer to the pair (X, σ) as a **real curve** or sometimes as a **Klein surface**.
- This gives rise to new **topological invariants** (besides the genus of X). Namely, the fixed-point set X^σ is a disjoint union of n circles with

$$0 \leqslant n \leqslant g + 1 .$$

- The topological surface $X/\langle\sigma\rangle$ may be non-orientable (set $a := 0$ or 1).
- Two real curves (X_1, σ_1) and (X_2, σ_2) are **equivariantly homeomorphic** if and only if $(g_1, n_1, a_1) = (g_2, n_2, a_2)$. This is also equivalent to saying that $X_1/\langle\sigma_1\rangle$ and $X_2/\langle\sigma_2\rangle$ are homeomorphic.

Real structure of the moduli space

- Let $\sigma : X \to X$ be a real structure on X. If $\mathcal{E}$ is a holomorphic vector bundle of rank r and degree d on X, then so is $\overline{\sigma^*\mathcal{E}}$. Moreover, if $\mathcal{E}$ is semistable, then so is $\overline{\sigma^*\mathcal{E}}$.
- If $0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_\ell = \mathcal{E}$ is the HN filtration of $\mathcal{E}$, then the HN filtration of $\overline{\sigma^*\mathcal{E}}$ is

$$0 = \overline{\sigma^*\mathcal{E}_0} \subsetneq \overline{\sigma^*\mathcal{E}_1} \subsetneq \cdots \subsetneq \overline{\sigma^*\mathcal{E}_\ell} = \overline{\sigma^*\mathcal{E}}.$$

- This shows two things:
  1. The moduli variety $\mathcal{M}_X(r, d)$ has an induced **real structure**.
  2. The HN strata $\mathscr{C}_\mu$ are stable under the **Galois action** $\mathcal{E} \mapsto \overline{\sigma^*\mathcal{E}}$.
- We are interested in the topology of the **real locus** $\mathbb{R}\mathcal{M}_X(r, d)$.

Moduli of real vector bundles

- To study $\mathbb{R}\mathcal{M}_X(r,d)$, we would like a **modular interpretation** of this real locus.
- For instance, if we have a real structure $\tau : \mathcal{E} \to \mathcal{E}$ on a (semistable) holomorphic vector $\mathcal{E}$ over X, compatible with σ, then $(\mathcal{E}, \tau)$ gives a real point of $\mathcal{M}_X(r,d)$. Such a vector bundle $(\mathcal{E}, \tau)$ is called a **real vector bundle** over (X, σ).
- But, due to the presence of non-trivial automorphisms for vector bundles, the converse is not true in general. Even for stable bundles, a real point of $\mathcal{M}_X(r,d)$ may come from $(\mathcal{E}, \tau)$ such that $\tau^2 = -Id_E$ instead of $\tau^2 = Id_E$. Such a vector bundle is called a **quaternionic vector bundle**.

Connected components of the real locus

- With this modular interpretation, we can use **real topological invariants** to count the connected components of $\mathbb{R}\mathcal{M}_X(r,d)$.
- Namely, for a real vector bundle $(\mathcal{E},\tau)$ over (X,σ), we get a vector bundle $\mathcal{E}^\tau \to X^\sigma$. This is a vector bundle whose fibre is a real vector space (in the usual sense) and X^σ is a disjoint union of circles, so $\mathcal{E}^\sigma$ is topologically classified by its first **Stiefel-Whitney class**, which consists of n mod 2 integers

$$w = (s_1, \ldots, s_n) \text{ such that } s_1 + \cdots + s_n = d \bmod 2.$$

- By previous results of S. (2012), when $r \wedge d = 1$ and $n > 0$, **the real variety** $\mathbb{R}\mathcal{M}_X(r,d)$ **has** 2^{n-1} **connected components**, which all consist of (real) isomorphism classes of geometrically stable real vector bundles of rank r and degree d. The invariant w distinguishes these connected components.
- When $n = 0$, we may get a quaternionic component (depending on r and d).

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Equivariant cohomology

- We compute the Poincaré series with **mod 2 coefficients** of the moduli stacks $[\mathscr{C}^\tau/\mathscr{G}_E^\tau]$ and $[\mathscr{C}_{ss}^\tau/\mathscr{G}_E^\tau]$ and, when $r \wedge d = 1$, the Poincaré polynomial of the moduli variety $\mathbb{R}\mathcal{M}_X(r,d)$.
- Since $\mathscr{C}^\tau$ is again an affine space (of Galois-invariant unitary connections),

$$P_t^{\mathscr{G}_E^\tau}(\mathscr{C}^\tau) = P_t(B\mathscr{G}_E^\tau)$$

  where $B\mathscr{G}_E^\tau$ is the classifying space of the **real gauge group** $\mathscr{G}_E^\tau$.
- This is the major difficulty in the real case: as we shall see, the formula for $P_t(B\mathscr{G}_E^\tau)$ is more involved, and the proof requires a case analysis, depending on the topological type of the Klein surface (X,σ).
- By analogy with Atiyah-Bott, we can prove that

$$B\mathscr{G}_E^\tau \sim \mathrm{Map}(X, B\mathbf{U}(r))^\tau.$$

The coprime case

- When $r \wedge d = 1$, $\mathscr{C}_s^\tau = \mathscr{C}_{ss}^\tau$ holds and $H^*(B\mathscr{G}_E^\tau; \mathbb{Z}/2\mathbb{Z}) \to H^*(B\mathbb{R}^*; \mathbb{Z}/2\mathbb{Z})$ is **surjective**.
- This will imply that, when X^σ has $n > 0$ connected components, we have

$$P_t(\mathbb{R}\mathcal{M}_X(r,d)) = 2^{n-1}(1-t)P_t^{\mathscr{G}_E^\tau}(\mathscr{C}_{ss}^\tau).$$

- So again the important part is to compute $P_t^{\mathscr{G}_E^\tau}(\mathscr{C}_{ss}^\tau)$ and this can be done recursively, if we know how to compute $P_t(B\mathscr{G}_E^\tau)$.

An equivariantly perfect stratification

- It follows from the uniqueness of the HN filtration that the HN filtration of a real vector bundle is a filtration by real sub-bundles. This shows that the Galois action on $\mathscr{C}$ preserves the HN strata. Hence a decomposition:

$$\mathscr{C}^{\tau} = \bigsqcup_{\mu \in I_{r,d}(\tau)} \mathscr{C}_{\mu}^{\tau}.$$

- We then proved that this stratification is **equivariantly perfect over mod 2 integers**. Hence:

$$P_t^{\mathscr{G}_E^{\tau}}(\mathscr{C}_{ss}^{\tau}) = P_t(B\mathscr{G}_E^{\tau}) - \sum_{\mu \in I_{r,d}(\tau) \mid d_{\mu} > 0} t^{d_{\mu}} P_t^{\mathscr{G}_E^{\tau}}(\mathscr{C}_{\mu}^{\tau}).$$

- Note that $\mathscr{C}_{\mu} \cap \mathscr{C}^{\tau}$ may be empty for some $\mu \in I_{r,d}$, so it is necessary to understand the set of **real HN types** $I_{r,d}(\tau)$.

The recursive step

- In this case, the initial step are components of the real part of the Picard variety, which all have Poincaré polynomial $(1+t)^g$. In particular, they do not depend on the real invariants of the real line bundles that they parameterize.
- If we can prove that $P_t(B\mathscr{G}_E^\tau)$ also does not depend on w, then by induction we will get that the $\mathscr{G}(r_i, d_i, w_i)$-equivariant Poincaré series of each $\mathscr{C}_{ss}^\tau(r_i, d_i, w_i)$ is **independent of** w_i.
- The recursive formula will then be written as follows:

$$P_t^{\mathscr{G}_E^\tau}(\mathscr{C}_{ss}^\tau) = P_t(B\mathscr{G}_E^\tau) - \sum_{\mu \in I_{r,d}(\tau) \mid d_\mu > 0} t^{d_\mu} \prod_{i=1}^{\ell} P_t^{\mathscr{G}(r_i,d_i)}(\mathscr{C}_{ss}^\tau(r_i, d_i)).$$

Cellular decompositions

- There remains to compute $P_t(B\mathscr{G}_E^{\tau})$, which is the Poincaré series of the stack of all holomorphic vector bundles of rank r and degree d.
- We want to start, as in the complex case, from a cellular decomposition of X, which this time should be compatible with the action of σ.
- This can be obtained from a cellular decomposition of $X \,/\, \langle\sigma\rangle$, but we have to consider several cases.
- For instance, when X^{σ} has $n > 0$ connected components, we have cellular decompositions of the form

$$\bigvee_{i=1}^{g+1} S^1 \hookrightarrow X \,/\, \langle\sigma\rangle \to S^2$$

for $X \,/\, \langle\sigma\rangle$, where the unique 0-cell x_1 is taken to lie *on the boundary*. The case $n = 0$ needs to be treated separately.

Two fibrations

- When we lift the celular decomposition to X, we get a cofibration

$$\vee_{i=1}^{2g+1} S^1 \hookrightarrow X \to S^2 \vee_{x_1} \sigma(S^2)$$

- By evaluating a Galois-equivariant map $f : X \to B\mathbf{U}(r)$ at the 0-cell $x_1 \in X^\sigma$, we get a fibration

$$\mathrm{Map}_d^\bullet(X, B\mathbf{U}(r))^\tau \to \mathrm{Map}_d(X, B\mathbf{U}(r))^\tau \to B\mathbf{O}(r).$$

- The second fibration comes from applying the pointed invariant-mapping functor to the cofibration above:

$$\mathrm{Map}_d^\bullet(S^2, B\mathbf{U}(r)) \to \mathrm{Map}_d^\bullet(X, B\mathbf{U}(r))^\tau \to \mathrm{Map}^\bullet(\vee_{i=1}^{g+1} S^1, B\mathbf{U}(r))^\tau.$$

- Note that $\mathrm{Map}_d^\bullet(S^2, B\mathbf{U}(r)) = \mathrm{Map}_d^\bullet(S^2 \vee_{x_1} \sigma(S^2), B\mathbf{U}(r))^\tau$.

Poincaré series

- We want to apply the Leray-Hirsch theorem to the two fibrations above, with mod 2 coefficients.
- There are two difficulties:
  1. $H^*(\Omega\mathbf{SU}(r);\mathbb{Z}/2\mathbb{Z})$ is not finitely generated as a $\mathbb{Z}/2\mathbb{Z}$-vector space.
  2. One has to prove, by case analysis on the topology of $X\,/\,\langle\sigma\rangle$, that $\mathrm{Map}^\bullet(\vee_{i=1}^{g+1}S^1, B\mathbf{U}(r))^\tau$ has the same mod 2 cohomology as

$$\mathbf{U}(r)^{g+1-n}\times\mathbf{SO}(r)^n\times(\mathbf{U}(r)/\mathbf{O}(r))^{n-1}.$$

- Eventually, this gives, when X^σ has $n>0$ connected components, the formula

$$\begin{aligned} P_t(B\mathscr{G}_E^\tau) &= P_t(\Omega\mathbf{SU}(r))P_t(\mathbf{U}(r))^{g+1-n}P_t(\mathbf{SO}(r))^nP_t(\mathbf{U}(r)/\mathbf{O}(r))^{n-1}P_t(B\mathbf{O}(r)) \\ &= \frac{\prod_{j=1}^r(1+t^{2j-1})^{g+1-n}\prod_{j=1}^{r-1}(1+t^j)^n\prod_{j=1}^r(1+t^j)^{n-1}}{\prod_{j=1}^{r-1}(1-t^{2j})\prod_{j=1}^r(1-t^j)}. \end{aligned}$$

Summary

When $n > 0$ we have constructed cohomologically trivial fibrations (over mod 2 integers):

$$\begin{array}{ccccc} \Omega\mathbf{SU}(r) & \longrightarrow & B\mathscr{G}_E^\tau(x_1) & \longrightarrow & \mathbf{U}(r)^{g+1-n} \times \mathbf{SO}(r)^n \times (\mathbf{U}(r)/\mathbf{O}(r))^{n-1} \\ & & \big\downarrow & & \\ & & B\mathscr{G}_E^\tau & & \\ & & \big\downarrow \mathrm{ev}_{x_1} & & \\ & & B\mathbf{O}(r) & & \end{array}$$

This contrasts with our previous approach, where we were using several base points in the cellular decomposition and the spectral sequences of the fibrations thus obtained did *not* collapse at the E_2-page.

Summary

Similarly, when $n = 0$ we can construct cohomologically trivial fibrations (over mod 2 integers):

$$\begin{array}{ccccc} \Omega \mathbf{SU}(r) & \longrightarrow & B\mathscr{G}_E^\tau(x_0) & \longrightarrow & \mathbf{U}(r)^{g+1} \\ & & \downarrow & & \\ & & B\mathscr{G}_E^\tau & & \\ & & \downarrow \mathrm{ev}_{x_0} & & \\ & & B\mathbf{U}(r) & & \end{array}$$

We note that the result for $P_t(B\mathscr{G}_E^\tau)$ can be expressed in a unified manner, because the fibration $\mathbf{U}(r)/\mathbf{O}(r) \to B\mathbf{O}(r) \to B\mathbf{U}(r)$ is trivial over mod 2 integers (Borel). So, for all $n \geqslant 0$,

$$P_t(B\mathscr{G}_E^\tau) = P_t(\Omega\mathbf{SU}(r))P_t(\mathbf{U}(r))^{g+1-n}P_t(\mathbf{SO}(r))^n P_t(\mathbf{U}(r)/\mathbf{O}(r))^n P_t(B\mathbf{U}(r)).$$