Real Seifert manifolds and the Narasimhan-Seshadri correspondence

Florent Schaffhauser

Universidad de Los Andes (Bogotá)

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Central extensions of the fundamental group

- M a compact connected Riemann surface of genus $g \geq 1$.
- Take $d \in \mathbb{Z} \simeq H^2(M;\mathbb{Z}) \simeq H^2(\pi_1(M);\mathbb{Z})$ and let us denote by

$$0 \longrightarrow \mathbb{Z} \longrightarrow \Gamma_d \longrightarrow \pi_1(M) \longrightarrow 1$$

  the corresponding central extension class.
- Let $\mathrm{Hom}^{\mathbb{Z}}(\Gamma_d; \mathbf{U}(r))$ consist of group homomorphisms $\rho : \Gamma_d \longrightarrow \mathbf{U}(r)$ such that, for all $n \in \mathbb{Z}$, $\rho(n) = \exp(i\frac{2\pi}{r}n) \in S_1$, i.e.:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \Gamma_d & \longrightarrow & \pi_1(M) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \exp(i\frac{2\pi}{r})} & & \downarrow{\scriptstyle \rho} & & \downarrow & & \\ 1 & \longrightarrow & S_1 & \longrightarrow & \mathbf{U}(r) & \longrightarrow & \mathbf{PU}(r) & \longrightarrow & 1 \end{array}$$

The Narasimhan and Seshadri theorem

Fix $g \geq 1$ and $d \in \mathbb{Z}$.

Theorem (Narasimhan and Seshadri, 1965)

For all $r \geq 1$, there is a homeomorphism

$$\mathcal{M}_{\mathbb{C}}^{\mathrm{ss}}(r, d) \simeq \mathbf{U}(r) \backslash \mathrm{Hom}^{\mathbb{Z}}(\Gamma_d; \mathbf{U}(r))$$

which restricts to a diffeomorphism

$$\mathcal{M}_{\mathbb{C}}^{\mathrm{s}}(r, d) \simeq \mathbf{U}(r) \backslash \mathrm{Hom}^{\mathbb{Z}}(\Gamma_d; \mathbf{U}(r))_{\mathrm{irr}}\,.$$

(*Caveat*: $\mathbf{U}(r) \backslash \mathcal{M}_{\mathbb{C}}^{\mathrm{s}}(r, d)$ and $\mathrm{Hom}^{\mathbb{Z}}(\Gamma_d; \mathbf{U}(r))_{\mathrm{irr}}$ are empty if $g = 1$ and $r \wedge d > 1$).
In this talk, we would like to explain an analogous statement for Real and Quaternionic vector bundles over Klein surfaces.

Klein surfaces

- A Klein surface is a pair (M, σ) where M is a Riemann surface and $\sigma : M \longrightarrow M$ is an anti-holomorphic involution. This defines a continuous action of $\Sigma = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \simeq \{1; \sigma\}$ on M.
- Weichold (1883): complete topological invariants (up to Σ-equivariant homeomorphism) of (M, σ) are given as follows.
  1. The genus g of M;
  2. The number $0 \leq n \leq g+1$ of connected components of M^Σ;
  3. The number $a = 0$ or 1, depending on the orientability of the quotient surface $\Sigma \backslash M$.
- The triple (g, n, a) will be called the topological type of (M, σ). It must satisfy:
  1. $(n = g+1) \Rightarrow (a = 0)$, $(n = 0) \Rightarrow (a = 1)$.
  2. $(a = 0) \Rightarrow (n \equiv (g+1) \mod 2)$.

Real and Quaternionic vector bundles

- Atiyah (1966), resp. Dupont (1969): a Real, resp. Quaternionic, vector bundle is a pair $(\mathcal{E}, \tau)$ consisting of a holomorphic vector bundle $p : \mathcal{E} \longrightarrow M$ and an anti-holomorphic map $\tau : \mathcal{E} \longrightarrow \mathcal{E}$ such that:
  1. The diagram
  $$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\tau} & \mathcal{E} \\ \downarrow p & & p \downarrow \\ M & \xrightarrow{\sigma} & M \end{array}$$
  is commutative.
  2. For all $\lambda \in C$ and all $v \in E$, $\tau(\lambda v) = \overline{\lambda} v$;
  3. $\tau^2 = \mathrm{Id}_{\mathcal{E}}$ (resp. $\tau^2 = -\mathrm{Id}_{\mathcal{E}}$).
- A homomorphism $f : (\mathcal{E}, \tau) \longrightarrow (\mathcal{E}', \tau')$ is a vector bundle homomorphism $f : \mathcal{E} \longrightarrow \mathcal{E}'$ (over Id_M) such that $f \circ \tau = \tau' \circ f$.

Goal of the talk

- Define moduli spaces $\mathcal{M}^{\mathrm{ss}}_{\mathbb{R}}(r,d)$ and $\mathcal{M}^{\mathrm{ss}}_{\mathbb{H}}(r,d)$ of semistable Real and Quaternionic vector bundles of rank r and degree d.
- The slope (semi-)stability condition on $(\mathcal{E},\tau)$ is defined with respect to τ-invariant sub-bundles $\mathcal{F}$: $\mu(\mathcal{F})(\leq)\mu(E)$.
- We want to construct a discrete group $\Gamma_d(\Sigma)$, a target Lie group $G_c(\Sigma)$, containing $\mathbf{U}(r)$, and an appropriate representation space $\mathrm{Rep}(\Gamma_d(\Sigma); G_c(\Sigma))$ such that the following Narasimhan and Seshadri correspondence holds:

$$\mathcal{M}^{\mathrm{ss}}_{c}(r,d) \simeq \mathrm{Rep}(\Gamma_d(\Sigma), G_c(\Sigma))$$

where $\mathcal{M}^{\mathrm{ss}}_{c}(r,d) = \begin{cases} \mathcal{M}^{\mathrm{ss}}_{\mathbb{R}}(r,d) & \text{if } c=+1, \\ \mathcal{M}^{\mathrm{ss}}_{\mathbb{H}}(r,d) & \text{if } c=-1. \end{cases}$

Categories of Real and Quaternionic vector bundles

- The Harder-Narasimhan filtration (1975) of a Real (resp. Quaternionic) vector bundle $(\mathcal{E}, \tau)$ consists of τ-invariant sub-bundles. In particular, any Real (resp. Quaternionic) bundle $(\mathcal{E}, \tau)$ is a successive extension of finitely many semistable Real (resp. Quaternionic) bundles.
- $(\mathcal{E}, \tau)$ is stable if and only if either $\mathcal{E}$ is stable or $\mathcal{E} = \mathcal{F} \oplus \sigma(\mathcal{F})$ with $\mathcal{F}$ stable and $\mathcal{F} \not\simeq \sigma(F) := \overline{(\sigma^{-1})^* \mathcal{F}}$.
- An analogue of Seshadri's 1967 result: by allowing stable Real (resp. Quaternionic) bundles that are only polystable as holomorphic bundles, we obtain the existence of Jordan-Hölder filtrations for semistable Real (resp. Quaternionic) bundles such that the graded objects associated to any two filtrations are isomorphic.
- The notion of polystable Real or Quaternionic vector bundle follows accordingly.

Moduli sets

$$\mathcal{M}_c^{\mathrm{ss}}(r,d) := \{\text{semistable } \mathbb{R} \text{ (or } \mathbb{H}) \text{ bundles}, \mathrm{rk}\, r, \deg d\}/S$$

- $(\mathcal{E}, \tau)$ is semistable/polystable as a Real (resp. Quaternionic) bundle if and only if $\mathcal{E}$ is semistable/polystable as a holomorphic bundle.
- As a consequence, there are maps

  $$\mathcal{M}_{\mathbb{R}}^{\mathrm{ss}}(r,d) \longrightarrow \mathcal{M}_{\mathbb{C}}^{\mathrm{ss}}(r,d)^{\Sigma}, \; \mathcal{M}_{\mathbb{H}}^{\mathrm{ss}}(r,d) \longrightarrow \mathcal{M}_{\mathbb{C}}^{\mathrm{ss}}(r,d)^{\Sigma},$$

  where Σ acts on $\mathcal{M}_{\mathbb{C}}^{\mathrm{ss}}(r,d)$ via $\sigma : [\mathcal{E}]_S \longmapsto [\overline{(\sigma^{-1})^*(\mathcal{E})}]_S$.
- Geometrically stable case: there is an exact sequence

  $$\bigsqcup_{c=\mathbb{R},\mathbb{H}} \mathcal{M}_c^{\mathrm{s}}(r,d) \longrightarrow \mathcal{M}_{\mathbb{C}}^{\mathrm{s}}(r,d)^{\Sigma} \xrightarrow{\mathcal{T}} H^2(\Sigma;\mathbb{C}^*) \simeq \{\mathbb{R};\mathbb{H}\}$$

  where $\mathbb{C}^* \simeq \mathrm{Aut}(\mathcal{E})$ for $\mathcal{E}$ stable. The first map is injective because $H^1(\Sigma;\mathbb{C}^*) = \{1\}$ (Hilbert's 90).

Topological classification

Theorem (Karoubi-Weibel 2003, Biswas-Huisman-Hurtubise 2010)

- $\tau = \tau_{\mathbb{R}}$, $M^{\Sigma} = \emptyset$: *There is a* τ*-compatible homeomorphism* $(\mathcal{E}, \tau) \simeq (\mathcal{E}', \tau')$ *if and only if* $(r, d) = (r', d')$*. Such bundles exist iff* $d = 2d'$*.*
- $\tau = \tau_{\mathbb{R}}$, $M^{\Sigma} \neq \emptyset$: *There is a* τ*-compatible homeomorphism* $(\mathcal{E}, \tau) \simeq (\mathcal{E}', \tau')$ *if and only if* $(r, d, w) = (r', d', w')$ *where* $w = w_1(E^{\tau}) = (s_1, \ldots, s_n) \in (\mathbb{Z}/2\mathbb{Z})^n$*. Such bundles exist iff* $s_1 + \ldots + s_n = d \mod 2$*.*
- $\tau = \tau_{\mathbb{H}}$: *There is a* τ*-compatible homeomorphism* $(\mathcal{E}, \tau) \simeq (\mathcal{E}', \tau')$ *if and only if* $(r, d) = (r', d')$*. Such bundles exist iff* $r = 2r'$ *and* $d = 2d'$ *(for* $M^{\Sigma} \neq \emptyset$*) and* $d + r(g-1) \equiv 0 \mod 2$ *(for* $M^{\Sigma} = \emptyset$*).*

Gauge theory

- Define auxiliary, refined moduli spaces (by topological type):
$$\mathcal{M}_c^{\mathrm{ss}}(r,d,w) := \begin{cases} \mathcal{M}_{\mathbb{R}}^{\mathrm{ss}}(r,d) & \text{if } c=+1 \text{ and } M^{\Sigma}=\emptyset; \\ \mathcal{M}_{\mathbb{R}}^{\mathrm{ss}}(r,d,w) & \text{if } c=+1 \text{ and } M^{\Sigma}\neq\emptyset; \\ \mathcal{M}_{\mathbb{H}}^{\mathrm{ss}}(r,d) & \text{if } c=-1. \end{cases}$$
- Point: $\mathcal{M}_c^{\mathrm{ss}}(r,d,w)$ embed into $\mathcal{M}_{\mathbb{C}}^{\mathrm{ss}}(r,d)^{\Sigma}$.
- (E,τ) a Real or Quaternionic $\mathcal{C}^{\infty}$ Hermitian vector bundle, $\mathcal{A}_E$ the space of unitary connections on E.
- Point: τ induces an involution β of $\mathcal{A}_E$ (regardless of whether $\tau^2 = \pm\mathrm{Id}_E$).
- $\mathcal{G}_E \subset \mathcal{G}_{\mathbb{C}}$: unitary and complex gauge groups of E.
  $\mathcal{G}_E^{\tau} \subset \mathcal{G}_{\mathbb{C}}^{\tau}$: Real parts (transformations that commute to τ).
- Point: $\mathcal{G}_{\mathbb{C}}^{\tau}$-orbits in $\mathcal{A}_E^{\tau}$ are in bijective correspondence with isomorphism classes of τ-compatible holomorphic structures on E.

Yang-Mills equations and the GIT picture

Theorem (2012)

Let (E, τ) be a Real or Quaternionic $\mathcal{C}^\infty$ Hermitian vector bundle on (M, σ). Then there is a homeomorphism

$$\mathcal{M}^{ss}(E, \tau) \simeq \mathcal{G}_E^\tau \backslash (F^{-1}(\{i(d/r)\}) \cap \mathcal{A}_E^\tau)$$

between the space of S-equivalence classes of τ-compatible, semistable holomorphic structures on E and gauge equivalence classes of Σ-invariant, minimal Yang-Mills (=projectively flat) connections.

Corollary

The moduli space $\mathcal{M}^{ss}(E, \tau)$ is connected.

The (orbifold) fundamental group of a Klein surface

- $\Sigma = \{1; \sigma\} \simeq \mathbb{Z}/2\mathbb{Z}$. $E\Sigma := S_\infty$, $B\Sigma := \Sigma \backslash S_\infty = \mathbb{R}\mathbf{P}_\infty$.
- $M_\Sigma := \Sigma\backslash(M \times E\Sigma)$, the homotopy quotient of the Σ-action on M. Since M is connected and Σ is discrete, the fibration $M \longrightarrow M_\Sigma \longrightarrow B\Sigma$ gives to a short exact sequence

$$1 \longrightarrow \pi_1(M) \longrightarrow \pi_1(M_\Sigma) \xrightarrow{\alpha} \Sigma \longrightarrow 1$$

  (only defined up to non-canonical isomorphism of Σ-augmentations if we do not fix a base point).
- $\pi_1(M_\Sigma)$ is an augmentation of the group Σ, meaning that the group homomorphism $\alpha : \pi_1(M_\Sigma) \longrightarrow \Sigma$ is surjective.
- Note that the group homomorphism $\Sigma \longrightarrow \mathrm{Out}(\pi_1(M))$ induced by the short exact sequence above can be defined geometrically (via $\sigma : M \longrightarrow M$) and actually determines $\pi_1(M_\Sigma; x)$ as an extension of Σ by $\pi_1(M; x)$ when $\mathcal{Z}(\pi_1(M; x)) = \{1\}$, *i.e.*, here, when $g \geq 2$.

Representations of orbifold fundamental groups

- A representation of $\pi_1(M_\Sigma)$ is a homomorphism of Σ-augmentations

$$\begin{array}{ccc} \pi_1(M_\Sigma) & \xrightarrow{\alpha} & \Sigma \\ \downarrow \chi & & \| \\ G(\Sigma) & \xrightarrow{\alpha_G} & \Sigma \end{array}$$

  In particular, χ is a group homomorphism and $\chi(\pi_1(M)) \subset G := \ker \alpha_G$.
- Two representations χ and χ' are called equivalent if there exists $g \in G$ such that $\chi' = \mathrm{Int}_g \circ \chi$.
- The associated representation space $G \backslash \mathrm{Hom}_\Sigma(\pi_1(M_\Sigma); G(\Sigma))$ maps, via $\chi \longmapsto \chi|_{\pi_1(M)}$, to the Σ-fixed-point set of the Σ-action defined on the usual representation variety $G \backslash \mathrm{Hom}(\pi_1(M); G)$ by $(\Sigma \longrightarrow \mathrm{Out}(G) \times \mathrm{Out}(\pi_1(M)))$.

Uniformization

- The universal cover of M_Σ is $\widetilde{M} \times E\Sigma$. Hence an isomorphism of Σ-augmentations
$$\pi_1(M_\Sigma) \xrightarrow{\simeq} \mathrm{Aut}((\widetilde{M} \times E\Sigma)/M_\Sigma).$$
- The group $\mathrm{Aut}((\widetilde{M} \times E\Sigma)/M_\Sigma)$ is isomorphic, as a Σ-augmentation, to
$$\mathrm{Aut}_\Sigma(\widetilde{M}/M) := \left\{ h : \widetilde{M} \longrightarrow \widetilde{M} \mid \exists\, \sigma_h \in \Sigma, \begin{array}{ccc} \widetilde{M} & \xrightarrow{h} & \widetilde{M} \\ \downarrow & & \downarrow \\ M & \xrightarrow{\sigma_h} & M \end{array} \right\}.$$

In particular, $\pi_1(M_\Sigma)$ acts on $\widetilde{M}$ (but Σ, in general, does not).

Real line bundles

- Kahn (1987): $\mathcal{C}^\infty$ Real line bundles are classified by their equivariant first Chern class

$$c_1^\Sigma(L, \tau_L) \in H^2_\Sigma(M; \underline{\mathbb{Z}})$$

  where Σ acts on $\mathbb{Z}$ via $n \longmapsto (-n)$.
- Let $(S(L), \tau_L)$ be the unit circle bundle in (L, τ_L). By definition, this is a Real Seifert manifold.
- $g \geq 1$: $H^2_\Sigma(M; \underline{\mathbb{Z}}) = H^2(M_\Sigma; \underline{\mathbb{Z}}) \simeq H^2(\pi_1(M_\Sigma); \mathbb{Z})$ where $\pi_1(M_\Sigma)$ acts on $\mathbb{Z}$ via $\alpha : \pi_1(M_\Sigma) \longrightarrow \Sigma$ and the above Σ-action on $\mathbb{Z}$. So $c_1^\Sigma(L, \tau_L)$ corresponds to an isomorphism class of (non-central) extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(S(L)_\Sigma) \longrightarrow \pi_1(M_\Sigma) \longrightarrow 1$$

  of $\pi_1(M_\Sigma)$ by $\mathbb{Z}$.

Extensions of $\pi_1(M_\Sigma)$

- By functoriality of the orbifold fundamental group, the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(S(L)) & \longrightarrow & \pi_1(S(L)_\Sigma) & \longrightarrow & \Sigma & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \pi_1(M) & \longrightarrow & \pi_1(M_\Sigma) & \longrightarrow & \Sigma & \longrightarrow & 1 \end{array}$$

  commutes.
- The map between the kernels is $\pi_1(S(L)) \longrightarrow \pi_1(M)$ which, by an observation due to Furuta and Steer (1992), coincides with the central extension Γ_d of $\pi_1(M)$ by $\mathbb{Z}$ determined by $d = c_1(L)$.
- $H^2(\Sigma; \mathcal{Z}(\pi_1(S(L)))) = \{0\}$, so the isomorphism class of $\Gamma_d(\Sigma) := \pi_1(S(L)_\Sigma)$ as a Σ-augmentation only depends on $d = c_1(L)$, not the full $c_1^\Sigma(L, \tau_L)$.

Enlarged unitary groups

- We want to construct an extension

$$1 \longrightarrow G \longrightarrow G_c(\Sigma) \longrightarrow \Sigma \longrightarrow 1$$

where $G = \mathbf{U}(r)$ and Σ acts on $\mathbf{U}(r)$ via $\sigma_{\mathbb{R}}(u) = \overline{u}$.

- We have $H^2(\Sigma; \mathcal{Z}(\mathbf{U}(r))) \simeq \{\pm 1\}$, so there are essentially two such extensions. Namely, for $c = \pm 1$:

$$1 \longrightarrow \mathbf{U}(r) \longrightarrow \mathbf{U}(r) \times_c \Sigma \longrightarrow \Sigma \longrightarrow 1, .$$

- When $c = +1$, $\mathbf{U}(r) \times_c \Sigma \simeq \mathbf{U}(r) \rtimes_{\sigma_{\mathbb{R}}} \Sigma$ as an extension of Σ by $\mathbf{U}(r)$.
- $c = +1$ will be used for Real bundles and $c = -1$ for Quaternionic ones. Note that $H^2(\Sigma; S_1) \simeq H^2(\mathrm{Gal}(\mathbb{C}/\mathbb{R}); \mathbb{C}^*) \simeq \mathrm{Br}(\mathbb{R}) \simeq \{\mathbb{R}; \mathbb{H}\}$.

Appropriate representations

- Denote by $\mathrm{Hom}^{\mathbb{Z}}_{\Sigma}(\Gamma_d(\Sigma); \mathbf{U}(r) \times_c \Sigma)$ the set of homomorphisms of Σ-augmentations

$$\chi : \Gamma_d(\Sigma) \longrightarrow \mathbf{U}(r) \times_c \Sigma$$

  such that, for all $n \in \mathbb{Z}$, $\chi(n) = \exp(i\frac{2\pi}{r}n) \in S_1$.
- The relevant representation space for us is

$$\mathbf{U}(r)\backslash\mathrm{Hom}^{\mathbb{Z}}_{\Sigma}(\Gamma_d(\Sigma); \mathbf{U}(r) \times_c \Sigma).$$

- When $d = 0$, $\Gamma_d(\Sigma) \simeq \mathbb{Z} \rtimes \pi_1(M_\Sigma)$ where $\pi_1(M_\Sigma)$ acts on $\mathbb{Z}$ via $\alpha : \pi_1(M_\Sigma) \longrightarrow \Sigma$ and $n \longmapsto (-n)$, so the representation space above is homeomorphic to

$$\mathbf{U}(r)\backslash\mathrm{Hom}_{\Sigma}(\pi_1(M_\Sigma); \mathbf{U}(r) \times_c \Sigma).$$

  When moreover $c = +1$, this coincides with the representation space of Biswas, Huisman, Hurtubise (2010).

Construction of the Narasimhan-Seshadri map 1

- We want to construct a map NS that takes

$$\chi \in \mathrm{Hom}_{\Sigma}^{\mathbb{Z}}(\Gamma_d(\Sigma); \mathbf{U}(r) \times_c \Sigma)$$

  to a pair $(\mathcal{E}, \tau)$ where $\mathcal{E}$ is a holomorphic vector bundle and τ is an anti-holomorphic map satisfying $\tau^2 = c = \pm 1$.
- Recall that there is a surjective map $\alpha_d : \Gamma_d(\Sigma) \longrightarrow \Sigma$ and choose $\widetilde{\sigma} \in \Gamma_d(\Sigma)$ above $\sigma \in \Sigma$. Two such choices differ by an element in $\ker \alpha_d = \Gamma_d$.
- $\Gamma_d(\Sigma)$ acts on $\widetilde{M}$ via the map $\Gamma_d(\Sigma) \longrightarrow \pi_1(M_\Sigma)$. Note that $\widetilde{\sigma}^2$ does not induce $\mathrm{Id}_{\widetilde{M}}$ in general.

Construction of the Narasimhan-Seshadri map 2

- Let $\rho := \chi|_{\Gamma_d}$. This defines a holomorphic vector bundle $\mathcal{E}_\rho := \Gamma_d \backslash (\widetilde{M} \times \mathbb{C}^r)$ of rank r and degree d on M.
- Define $u_{\widetilde{\sigma}} \in \mathbf{U}(r)$ by the equation $\chi(\widetilde{\sigma}) = (u_{\widetilde{\sigma}}, \sigma)$ and consider the anti-holomorphic transformation

$$\widetilde{\tau} : \begin{array}{ccc} \widetilde{M} \times \mathbb{C}^r & \longrightarrow & \widetilde{M} \times \mathbb{C}^r \\ (\delta, v) & \longmapsto & (\widetilde{\sigma} \cdot \delta, u_{\widetilde{\sigma}} \overline{v}) \end{array}.$$

Construction of the Narasimhan-Seshadri map 3

Proposition

Given $\chi \in \mathrm{Hom}^{\mathbb{Z}}_{\Sigma}(\Gamma_d(\Sigma); \mathbf{U}(r) \times_c \Sigma)$, denote by ρ the representation $\chi|_{\Gamma_d}$. Then the map $\widetilde{\tau}$ defined earlier induces a Real structure τ on $\mathcal{E}_\rho$ if $c = +1$ and a Quaternionic one if $c = -1$. Moreover:

- *Equivalent representations χ and χ' give rise to isomorphic Real or Quaternionic bundles.*
- *A different choice of $\widetilde{\sigma}$ gives rise to a Real or Quaternionic structure τ' which is conjugate to τ.*

By the Narasimhan and Seshadri theorem, $\mathcal{E}_\rho$ is polystable as a holomorphic vector bundle and we know that this implies that $(\mathcal{E}_\rho, \tau)$ is polystable as a Real or Quaternionic vector bundle.

The Narasimhan and Seshadri correspondence

Theorem

The Narasimhan and Seshadri map

$$\mathbf{U}(r)\backslash \mathrm{Hom}^{\mathbb{Z}}_{\Sigma}(\Gamma_d(\Sigma); \mathbf{U}(r) \times_c \Sigma) \xrightarrow{\mathrm{NS}} \mathcal{M}^{\mathrm{ss}}_c(r,d)$$

is a homeomorphism, i.e. any polystable Real or Quaternionic vector bundle is isomorphic to a bundle of the form $(\mathcal{E}_\rho, \tau)$ constructed as ealier from a representation of $\Gamma_d(\Sigma)$.

Corollary

The homeomorphism type of the moduli spaces $\mathcal{M}^{\mathrm{ss}}_{\mathbb{R}}(r,d)$ and $\mathcal{M}^{\mathrm{ss}}_{\mathbb{H}}(r,d)$ is independent of the complex structure on M.

Sketch of the proof

- Decompose $\mathcal{M}_c^{\mathrm{ss}}(r,d)$ into

$$\bigsqcup_w \mathcal{M}_c^{\mathrm{ss}}(r,d,w) = \bigsqcup_{(E,\tau)} \mathcal{G}_E^\tau \backslash (F^{-1}(\{i(d/r)\}) \cap \mathcal{A}_E^\tau).$$

- Construct, for all (E,τ), a holonomy map

$$\mathcal{G}_E^\tau \backslash (F^{-1}(\{i(d/r)\}) \cap \mathcal{A}_E^\tau) \longrightarrow \mathbf{U}(r) \backslash \mathrm{Hom}_\Sigma^{\mathbb{Z}}(\Gamma_d(\Sigma); \mathbf{U}(r) \times_c \Sigma)$$

and show that the collection of such maps provides an inverse to the NS map.

Galois-invariant connections

- Let A be a Σ-invariant connection on (E, τ). Choose a base point $x \in M$ and a frame $\varphi : E_x \xrightarrow{\simeq} \mathbb{C}^r$ at x.
- Given a pair $(\gamma, \lambda) \in \pi_1(M) \times \Sigma$, where γ is a path from x to $\lambda^{-1}(x)$, define $g_\lambda \in \mathbf{U}(r)$ by the condition

$$(\tau_\lambda \circ T_\gamma^A)(v) = \begin{cases} g_\lambda v & \text{if } \lambda = 1, \\ g_\lambda \overline{v} & \text{if } \lambda = \sigma \end{cases}$$

  where T_γ^A is the parallel transport along γ associated to A.
- In other words, g_λ is the matrix of the λ-linear map $\tau_\lambda \circ T_\gamma^A : E_x \longrightarrow E_x$ in the frame φ.

Enlarged holonomy group

- $\mathrm{Hol}_x^{\Sigma}(A) := \{(g_\lambda, \lambda) \in \mathbf{U}(r) \times_c \Sigma\}$ is a sub-Σ-augmentation of $\mathbf{U}(r) \times_c \Sigma$.
- The map $\widetilde{\chi} : (\gamma, \lambda) \longmapsto (g_\lambda, \lambda)$ induces a homomorphism of Σ-augmentations

$$\overline{\chi} : \pi_1(M_\Sigma) \longrightarrow \mathbf{PU}(r) \rtimes \Sigma$$

which in turn induces a homomorphism of Σ-augmentations

$$\chi : \Gamma_d(\Sigma) \longrightarrow \mathbf{U}(r) \rtimes \Sigma$$

satisfying, for all $n \in \mathbb{Z}$, $\chi(n) = \exp(i\frac{2\pi}{r}n)$.

Klein surfaces with Real points

- Recall that we have two extensions $\mathbf{U}(r) \times_c \Sigma$, $c = \pm 1$. When $c = +1$, this is the semi-direct product $\mathbf{U}(r) \rtimes_{\sigma_{\mathbb{R}}} \Sigma$ where $\sigma_{\mathbb{R}}(u) = \overline{u}$.
- Recall that, over a Klein surface with Real points, Quaternionic vector bundles must have even rank $r = 2r'$, say.
- When $c = -1$ and $r = 2r'$, $\mathbf{U}(r) \times_c \Sigma$ is isomorphic to $\mathbf{U}(2r') \rtimes_{\sigma_{\mathbb{H}}} \Sigma$ where $\sigma_{\mathbb{H}}(u) = J\overline{u}J^{-1}$.
  Proof: The $\mathcal{Z}(G)$-cocycle $c = -I_r$ "splits over G": $c = J^2 = J\sigma(J)$.
- Note that $\sigma_{\mathbb{H}}$ induces the same outer action as $\sigma_{\mathbb{R}}$, as well as the same action on $\mathcal{Z}(\mathbf{U}(r)) \simeq S_1$.

Equivariant representations

- Fact: If Σ acts on $G = \mathbf{U}(r)$ via $\sigma_{\mathbb{R}}$ or $\sigma_{\mathbb{H}}$, one has $H^1(\Sigma; G) = \{1\}$.
- As a consequence,

$$G \backslash \mathrm{Hom}_{\Sigma}(\Gamma \rtimes \Sigma; G \rtimes \Sigma) \simeq G^{\Sigma} \backslash \mathrm{Hom}(\Gamma; G)^{\Sigma}.$$

- Note that, in general,

$$G \backslash \mathrm{Hom}_{\Sigma}(\Gamma \rtimes \Sigma; G \rtimes \Sigma) \simeq \bigsqcup_{[a] \in H^1(\Sigma; G)} G^{\Sigma_a} \backslash \mathrm{Hom}(\Gamma; G)^{\Sigma_a}$$

where Σ_a means that σ acts on G via $\sigma \cdot g = a_\sigma \sigma(g) a_\sigma^{-1}$.

The Narasimhan and Seshadri correspondence when $M^\Sigma \neq \emptyset$

Theorem

If $M^\Sigma \neq \emptyset$, then there is an action of Σ on Γ_d and one has

$$\mathcal{M}_{\mathbb{R}}^{\mathrm{ss}}(r,d) \simeq \mathbf{O}(r)\backslash \mathrm{Hom}(\Gamma_d;\mathbf{U}(r))^{\sigma_{\mathbb{R}}}$$

and

$$\mathcal{M}_{\mathbb{H}}^{\mathrm{ss}}(r,d) \simeq \mathbf{Sp}(r/2)\backslash \mathrm{Hom}(\Gamma_d;\mathbf{U}(r))^{\sigma_{\mathbb{H}}}.$$

- The NS map can be constructed more directly when $M^\Sigma \neq \emptyset$, because Σ acts on $\widetilde{M}$ in this case (i.e. one has $\widetilde{\sigma}^2 = 1$). Just consider the maps $\widetilde{\tau} : (\delta, v) \longmapsto (\widetilde{\sigma}(\delta), \overline{v})$ and $\widetilde{\tau} : (\delta, v) \longmapsto (\widetilde{\sigma}(\delta), J\overline{v})$ on $\widetilde{M} \times \mathbb{C}^r$.
- One may note that the classical NS map is always Σ-equivariant but that it is complicated to analyze the fixed-point sets $\mathcal{M}_{\mathbb{C}}^{\mathrm{ss}}(r,d)^\Sigma \simeq (\mathbf{U}(r)\backslash \mathrm{Hom}^{\mathbb{Z}}(\Gamma_d;\mathbf{U}(r)))^\Sigma$.

Obstruction maps

- When $c = +1$ and $M^\Sigma \neq \emptyset$, there is a well-defined map

$$\mathcal{W} : \begin{array}{ccc} \mathbf{O}(r)\backslash \mathrm{Hom}^{\mathbb{Z}}(\Gamma_d(\Sigma); \mathbf{U}(r))^\Sigma & \longrightarrow & \mathbf{O}(1)^n \\ \rho & \longmapsto & (\det \rho(\alpha_1))_{1 \leq i \leq n} \end{array}$$

  where the α_i are essentially the loops $\gamma_1 \sqcup \ldots \sqcup \gamma_n = M^\Sigma$.
- The NS map establishes a homeomorphism

$$\mathcal{M}_c^{\mathrm{ss}}(r, d, w) \xrightarrow{\mathrm{NS}} \mathcal{W}^{-1}(w)$$

  so the connected components of $\mathbf{O}(r)\backslash \mathrm{Hom}^{\mathbb{Z}}(\Gamma_d; \mathbf{U}(r))^\Sigma$ are precisely the fibers of $\mathcal{W}$.
- In contrast, the connected components of

$$\Big(\mathbf{U}(r)\backslash \mathrm{Hom}^{\mathbb{Z}}(\Gamma_d; \mathbf{U}(r))\Big)^\Sigma$$

  are not known in general.