# Hodge numbers of moduli stacks of principal bundles

$$
\begin{tikzcd}
\Omega\widetilde{G_{ss}} \arrow[r] & \mathrm{Bun}_{G_X}^{d,\bullet} \arrow[r, red] \arrow[d] & G^{2g_X} \\
& \mathrm{Bun}_{G_X}^{d} \arrow[d] & \\
& BG &
\end{tikzcd}
$$

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# G-bundles on algebraic curves

$X =$  [smooth, projective $/\mathbb{C}$]

$g = g_X = \dim H^0(X; \Omega^1_X)$ [genus]

$G = GL(r;\mathbb{C}),\ SL(r;\mathbb{C}),\ SO(2r+1;\mathbb{C}),\ \dots$

$\rightsquigarrow$ connected, reductive algebraic group $/\mathbb{C}$

$$\mathrm{Bun}_{G_X} : \left(Sch/\mathbb{C}\right)^{op} \longrightarrow \text{groupoids}$$

$$S \longmapsto \langle \underbrace{E \to S \times X}_{S\text{-family of } G\text{-bundles on } X} \rangle$$

# Poincaré series of the moduli stack of vector bundles

<u>Theorem</u> (Harder–Narasimhan 1975, Atiyah–Bott 1983)

For all $d \in \mathbb{Z}$,

$$\boxed{P_t(\mathrm{Vect}_X(r,d);\mathbb{Q}) = \frac{(1+t)^{2g_X}}{1-t^2} \prod_{k=2}^{r} \frac{(1+t^{2k-1})^{2g_X}}{(1-t^{2k-2})(1-t^{2k})}}$$

$$H^*(\mathrm{Vect}_X(r,d)) \simeq H^*(\Omega SL(r;\mathbb{C})) \otimes H^*(BGL(r;\mathbb{C})) \otimes H^*(GL(r;\mathbb{C}))^{2g_X}$$

$$\frac{1}{\prod_{k=2}^{r}(1-t^{2k-2})} \times \frac{1}{\prod_{k=1}^{r}(1-t^{2k})} \times \prod_{k=1}^{r}(1+t^{2k-1})^{2g_X}$$

## Hodge-Poincaré series

**Theorem** (Earl - Kirwan 2000) [+ Teleman 1998]

$G = SL(r;\mathbb{C}), \ldots$

For all $d \in \mathbb{Z}$,

$$\boxed{HP_{u,v}(\mathrm{Vect}_X(r,d)) = \frac{(1+u)^{g_X}(1+v)^{g_X}}{1-uv} \overbrace{\prod_{k=2}^{r} \frac{(1+u^k v^{k-1})^{g_X}(1+u^{k-1}v^k)^{g_X}}{(1-(uv)^{k-1})(1-(uv)^k)}}}$$

$\rightsquigarrow$ What about stacks of $G$-bundles for other $G$ ?

## Hodge structures

[pure, over $\mathbb{Q}$]

$V_{\mathbb{Q}}$ : a $\mathbb{Q}$-vector space

$(V^{p,q})_{p,q\geqslant 0}$ : finite-dimensional $\mathbb{C}$-vector spaces

$\varphi$ : an isomorphism of $\mathbb{C}$-vector spaces

$$\bigoplus_{k=0}^{+\infty} \bigoplus_{p+q=k} V^{p,q} \xrightarrow[\varphi]{\simeq} V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$$

such that $\overline{V^{p,q}} = V^{q,p}$ as subspaces of $V_{\mathbb{C}} := V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$

Hodge series : $$HP_{u,v}(V_{\mathbb{C}}) = \sum_{k=0}^{+\infty} \sum_{p+q=k} u^p v^q \underbrace{\dim_{\mathbb{C}} V^{p,q}}_{<+\infty}$$

# Cohomology algebras

- $X$ smooth projective variety $/\mathbb{C}$ :

  $H^*(X;\mathbb{C})$ has a Hodge structure,

[Hodge decomposition]

  since $H^k(X;\mathbb{C}) \simeq \bigoplus_{p+q=k} \underbrace{H^q(X;\Omega^p_X)}_{H^{p,q}(X)}$

- $G$ : connected, reductive algebraic group $/\mathbb{C}$

  $BG = [pt/G]$ the classifying stack of $G$

[Deligne 1974]

  $H^*(BG;\mathbb{C})$ has a Hodge structure

# Hodge-proper stacks

[Kubrak - Prikhodko 2019]

$\mathcal{M}$ : a smooth algebraic stack $/ \mathbb{C}$

[Totaro 2018] $H^{p,q}(\mathcal{M}) := H^q(\mathcal{M}; \Lambda^p \underbrace{L_{\mathcal{M}}}_{\text{cotangent complex}})$

$$H^k(\mathcal{M}; \mathbb{C}) \underset{\text{De Rham}}{\simeq} H^k_{DR}(\mathcal{M}) \simeq \bigoplus_{p+q=k} H^{p,q}(\mathcal{M})$$

The notion of Hodge-proper stack gives a sufficient condition on $\mathcal{M}$ for this to happen $\Bigg\}$ The Hodge-to-De-Rham spectral sequence degenerates at the $E_1$-page

# Examples

① $H \simeq (\mathbb{C}^*)^r$ $\quad BH \simeq (\mathbb{C}P^\infty)^r$

$$H^*(BH;\mathbb{C}) \simeq \mathbb{C}[x_1, \dots, x_r]$$

$x_i \in H^{1,1}(BH)$

$$HP_{u,v}(BH) = \frac{1}{(1-uv)^r} \in \text{"}\mathbb{Q}(u,v) \cap \mathbb{Q}[[u,v]]\text{"}$$

② $G$ reductive, $H \subset G$ maximal torus, $W$ Weyl group

$$H^*(BG;\mathbb{C}) \xhookrightarrow{\sim} H^*(BH;\mathbb{C})^W \subset H^*(BH;\mathbb{C})$$

$$\mathbb{C}[I_1, \dots, I_r] \cong H^*(BH;\mathbb{C})^W$$

$I_k$ : homogeneous polynomial of degree $d_k$ in $(x_1, \dots, x_r)$

$I_k \in H^{d_k, d_k}(BG)$

# Hodge series of BG

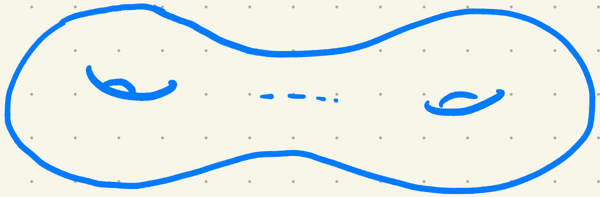
$$\underbrace{d_1 = \dots = d_m = 1}_{m = \dim Z_G} < d_{m+1} \leq \dots \leq d_r \qquad \text{("exponents of } G\text{")}$$

$$HP_{u,v}(BG) = \frac{1}{(1-uv)^m \prod_{k=m+1}^{r} (1-(uv)^{d_k})}$$

**Example:** $G = GL(r;\mathbb{C})$, $W = \mathfrak{S}_r$, $m = 1$, $\underbrace{d_k = k}_{2 \leq k \leq r}$

$$H^*(BGL(r;\mathbb{C});\mathbb{C}) \simeq \mathbb{C}[\underbrace{c_1, \dots, c_r}_{\text{universal Chern classes}}]$$

# Topological classification of $G$-bundles

$X =$  [smooth, projective / $\mathbb{C}$]

$E \to X$ a principal $G$-bundle

$\dim_{\mathbb{R}} X = 2 \rightsquigarrow 2$ obstruction classes

$o_1(E) \in H^1(X; \pi_0 G) \rightsquigarrow$ trivial if $G$ connected

$\deg(E) \leftsquigarrow o_2(E) \in H^2(X; \pi_1 G) \simeq \pi_1 G$ ($X$ oriented)

<u>Example</u>

$G = GL(r;\mathbb{C})$ $\quad o_2(E) = c_1(E) \in H^2(X;\mathbb{Z}) \simeq \mathbb{Z}.$

$G = SO(r;\mathbb{C})$ $\quad o_2(E) = w_2(E) \in H^2(X;\mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}.$

# Components of the moduli stack

$$\mathrm{Bun}_{G_X} = \coprod_{d \in \pi_1 G} \underbrace{\mathrm{Bun}_{G_X}^d}_{\text{irreducible open and closed substacks}}$$

$S \in \mathrm{Sch}/_{\mathbb{C}}$ :

$$\mathrm{Bun}_{G_X}^d(S) = \langle \mathbb{E} \to S \times X \mid \forall s \in S(\mathbb{C}),\ \deg \mathbb{E}_s = d \rangle$$

$\rightsquigarrow$ $X \times \mathrm{Bun}_{G_X}^d$ carries a universal bundle

$$\begin{tikzcd}
U_d \arrow[r] \arrow[d] & U \arrow[r] \arrow[d, "\text{``}id_{\mathrm{Bun}_{G_X}}\text{''}"] & EG \arrow[d] \\
\mathrm{Bun}_{G_X}^d \times X \arrow[r, hook] & \mathrm{Bun}_{G_X} \times X \arrow[r, "ev"] & BG
\end{tikzcd}$$

## Hodge series of the components

**Theorem** (Liu - S.)

For all $d \in \pi_1 G$,

$$HP_{u,v}\left(\mathrm{Bun}_{G_x}^d\right) = \left(\frac{(1+u)^{g_x}(1+v)^{g_x}}{1-uv}\right)^m \prod_{k=m+1}^{r} \frac{\left(1+u^{d_k}v^{d_k-1}\right)^{g_x}\left(1+u^{d_k-1}v^{d_k}\right)^{g_x}}{\left(1-(uv)^{d_k-1}\right)\left(1-(uv)^{d_k}\right)}$$

**Remarks**

(i) This confirms a conjecture of Behrend and Dhillon (2007), obtained by a motivic approach.

(ii) $HP_{u,v}\left(\mathrm{Bun}_{G_x}^d\right)$ is seen to be independent of $d$.

(iii) $m = 0$ : Teleman (1998), using algebraic loop groups.

# Ingredients of the proof

(i) Hodge theory of the base curve $X$.

(ii) Atiyah - Bott generators of $H^*(\mathrm{Bun}^d_{G_x}; \mathbb{C})$.

(ii) Recall that

$$H^*(BG;\mathbb{C}) \simeq \mathbb{C}[I_{d_1}, \dots, I_{d_r}]$$

where $I_k \in H^{d_k, d_k}(BG;\mathbb{C})$ is a universal characteristic class.

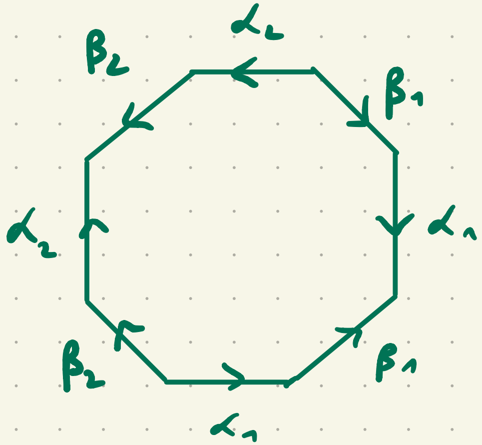
So the universal bundle $U_d \to X \times \mathrm{Bun}^d_{G_x}$ defines cohomology classes

$$\underbrace{I_k(U_d)}_{:= ev^* I_k} \in H^{2k_k}(\mathrm{Bun}^d_{G_x} \times X; \mathbb{C}).$$

, where $ev : X \times \mathrm{Bun}^d_{G_x} \longrightarrow BG$.

# Künneth decomposition

$$H^{2d_k}(\mathrm{Bun}^d_{G_X}\times X;\mathbb{C}) \simeq H^{2d_k}(\mathrm{Bun}^d_{G_X};\mathbb{C}) \otimes H^0(X;\mathbb{C})$$




$$\oplus\ H^{2d_k-1}(\mathrm{Bun}^d_{G_X};\mathbb{C}) \otimes H^1(X;\mathbb{C})$$

$$\oplus\ H^{2d_k-2}(\mathrm{Bun}^d_{G_X};\mathbb{C}) \otimes H^2(X;\mathbb{C})$$

so

$$I_k(U) = h_k \otimes 1 + \sum_{j=1}^{g_X}\left(a^j_k \otimes \alpha_j^* + b^j_k \otimes \beta_j^*\right) + f_k \otimes \omega$$

$$H^{2d_k}(\mathrm{Bun}^d_{G_X};\mathbb{C}) \qquad H^{2d_k-1}(\mathrm{Bun}^d_{G_X};\mathbb{C}) \qquad H^{2d_k-2}(\mathrm{Bun}^d_{G_X};\mathbb{C})$$

$$\begin{tikzcd} \Omega\widetilde{G_{ss}} \arrow[r] & \mathrm{Bun}_{G_X}^{d,\bullet} \arrow[r, red, "\text{holonomy}"'] \arrow[d] & G^{2g_X} \\ & \mathrm{Bun}_{G_X}^{d} \arrow[d] & \\ & BG & \end{tikzcd}$$

# Applying Leray-Hirsch

**Theorem** (Atiyah-Bott, 1983)

$$H^*(\mathrm{Bun}_{G_X}^d; \mathbb{C}) \simeq \mathbb{C}[h_1, \dots, h_r, f_{m+1}, \dots, f_r] \otimes_{\mathbb{C}} \Lambda_{\mathbb{C}}\left[(a_i^j, b_i^j)_{1\leq j\leq g_X,\, 1\leq i\leq r}\right]$$

[hence the Poincaré series]

$\rightsquigarrow$ depends on a choice of basis for $H^1(X;\mathbb{C})$.

# Hodge classes

$\rightsquigarrow$ need a different basis for $H^1(X;\mathbb{C})$:

$$H^1(X;\mathbb{C}) \simeq \underbrace{\underbrace{H^0(X;\Omega^1_X)}_{\bigoplus_{j=1}^{g_X}\mathbb{C}\,\omega_j}}_{(1,0)\text{-classes}} \oplus \underbrace{\underbrace{H^1(X;\Omega^0_X)}_{\bigoplus_{j=1}^{g_X}\mathbb{C}\,\overline{\omega_j}}}_{(0,1)\text{-classes}}$$

Choose $\omega_i$ such that:

$$\int_{\alpha_j}\omega_i = \delta_{ij}$$

The two bases $(\alpha_j^*, \beta_j^*)_{1\le j\le g_X}$ and $(\omega_j, \overline{\omega_j})_{1\le j\le g_X}$ are related via the period matrix of $\Big(X, \underbrace{(\alpha_j,\beta_j)_{1\le j\le g_X}}_{\text{symplectic basis of } H_1(X;\mathbb{C})}\Big)$: $\left(\tau_{ij} = \int_{\beta_j}\omega_i\right)_{1\le i\le g_X}$.

## New generators

$$\underset{(d_t,\,d_t)}{I_t(U)} = \underset{\substack{\swarrow \\ H^{d_t,d_t}(\mathrm{Bun}^d_{G_X};\,\mathbb{C})}}{h_t} \otimes 1 + \sum_{j=1}^{g_X} \Big( \underset{\substack{\downarrow \\ H^{d_t-1,d_t}(\mathrm{Bun}^d_{G_X})}}{\theta_t^j} \otimes \overset{(1,0)}{\omega_j} + \underset{\substack{\downarrow \\ H^{d_t,d_t-1}(\mathrm{Bun}^d_{G_X})}}{\overline{\theta_t^j}} \otimes \overset{(0,1)}{\overline{\omega_j}} \Big) + \underset{\substack{\searrow \\ H^{d_t-1,d_t-1}(\mathrm{Bun}^d_{G_X};\,\mathbb{C})}}{f_t} \otimes \overset{(1,1)}{\omega}$$

and

$$\mathrm{span}_{\mathbb{C}}\left( a_t^j, b_t^j : 1 \le j \le g_X \right) = \mathrm{span}_{\mathbb{C}}\left( \theta_t^j, \overline{\theta_t^j} : 1 \le j \le g_X \right)$$

so

$$\boxed{H^*(\mathrm{Bun}^d_{G_X};\,\mathbb{C}) \simeq \mathbb{C}[h_1, \dots, h_r, f_{m+1}, \dots, f_r] \otimes_{\mathbb{C}} \Lambda_{\mathbb{C}}\left[ (\theta_t^j, \overline{\theta_t^j})_{1 \le j \le g_X,\, 1 \le t \le r} \right].}$$

Hence the Hodge series.

# Odd colomology

Evidently, $H^{odd}(\mathrm{Bun}^{d}_{G_x};\mathbb{C}) \neq 0$. But more precisely, $H^{p,q}(\mathrm{Bun}^{d}_{G_x}) \neq 0 \not\Rightarrow p=q$.

In contrast, $H^{p,q}(BG) \neq 0 \Rightarrow p=q$.

$\rightsquigarrow$ it is the odd cohomology of $X$ which, in view of the Künneth decomposition, causes classes in $H^{2d_\xi-1}(G;\mathbb{C})$ to pull back to Hodge classes of type $(d_\xi-1, d_\xi)$ and $(d_\xi, d_\xi-1)$ in $H^*(\mathrm{Bun}^{d}_{G_x};\mathbb{C})$.

## Poincaré series

For $u = v = t$, the Hodge series

$$HP_{u,v}\left(\mathrm{Bun}^d_{G_x}\right) = \left(\frac{(1+u)^{g_x}(1+v)^{g_x}}{1-uv}\right)^m \prod_{\ell=m+1}^{r} \frac{\left(1+u^{d_\ell}v^{d_\ell-1}\right)^{g_x}\left(1+u^{d_\ell-1}v^{d_\ell}\right)^{g_x}}{\left(1-(uv)^{d_\ell-1}\right)\left(1-(uv)^{d_\ell}\right)}$$

specializes to the Poincaré series

$$\boxed{P_t\left(\mathrm{Bun}^d_{G_x}\right) = \left(\frac{(1+t)^{2g_x}}{1-uv}\right)^m \prod_{\ell=m+1}^{r} \frac{1+t^{2d_\ell-1}}{\left(1-t^{2d_\ell-2}\right)\left(1-t^{2d_\ell}\right)}}$$

[Laumon-Rapoport, 1996]

# Real structures

$X =$ 


[smooth, projective $/\mathbb{R}$]

a maximal curve $(n_X = g_X + 1)$

**Theorem** Over $(X, \tau)$ maximal and for $G = GL(r; \mathbb{C})$

The Hodge series of $Bun^d_{G_X}$ specializes, for $u = t$ and $v = 1$, to the mod 2 Poincaré series of the substack of real points $\mathbb{R}Bun^d_{G_X}$:

in the sense of Romagny (2005)

$$HP_{t,1}\left(Bun^d_{G_X}\right) = 2^{g_X} \underbrace{\frac{(1+t)^{g_X}}{1-t} \prod_{k=2}^{r} \frac{(1+t^{k-1})^{g_X}(1+t^k)^{g_X}}{(1-t^{k-1})(1-t^k)}}_{= P_t(\mathbb{R}Bun^d_{G_X};\, \mathbb{Z}/2\mathbb{Z}) \text{ by Liu-S. (2013)}}$$

# of connected components of $\mathbb{R}Bun^d_{G_X}$ by S. (2012)

## Maximal varieties

A real variety $(X, \tau)$ is called maximal if the Smith inequality

$$P_t(\mathbb{R}X; \mathbb{Z}/2\mathbb{Z})\Big|_{t=1} \leq P_t(X; \mathbb{Z}/2\mathbb{Z})\Big|_{t=1}$$

is an equality, i.e.

$$b_0(\mathbb{R}X) + \dots + b_n(\mathbb{R}X) = b_0(X) + \dots + b_{2n}(X). \quad [n = \dim X]$$

**Definition** (Brugallé - S., 2022)

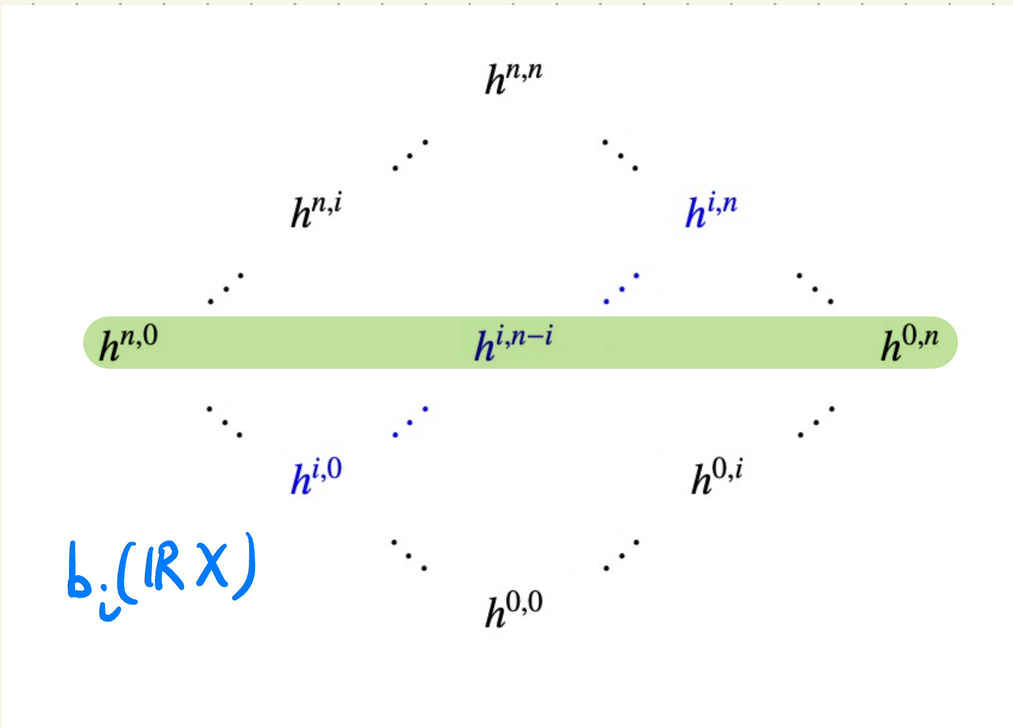
A smooth, projective real variety $(X, \tau)$ is called Hodge-expressive if:

(i) $H^*(X; \mathbb{Z})$ is torsion-free.

(ii) $P_t(\mathbb{R}X; \mathbb{Z}/2\mathbb{Z}) = HP_{t,1}(X)$.

# Hodge-expressive varieties are maximal

$b_n(X)$




$b_i(\mathbb{R}X)$

$\forall i \geq 0,$

$$b_i(\mathbb{R}X) = \sum_{j \geq 0} h^{i,j}(X)$$

**Remark** (Brugallé)

For such varieties, $\chi(\mathbb{R}X) = \sigma(X)$.

$\left[\text{since } \chi(\mathbb{R}X) = P_t(\mathbb{R}X)\big|_{t=-1} = HP_{t,1}(X)\big|_{t=-1} = \sigma(X)\right]$

# Maximality of moduli spaces of vector bundles

**Theorem** (Brugallé - S., 2022)

If $(X,\tau)$ is maximal and $r \wedge d = 1$, then the moduli space of semistable vector bundles of rank $r$ and degree $d$ is Hodge-expressive, hence also maximal.

$\rightsquigarrow$ new proof of this now follows from joint work with Lin

[closed formula for $HP_{u,v}(\mathrm{Bun}_{G_X}^{d,ss})$]

## Semistability

**Definition** (Ramanathan, 1975)

A principal $G$-bundle $E \to X$ is called **semistable** if, for all maximal parabolic subgroup $P \subset G$ and all reduction of structure group $\sigma : X \to E/P$,

$$\deg\left(\sigma^* E(\mathfrak{g}/\mathfrak{p})\right) \geq 0 .$$

Equivalently, if $E_P$ is the reduction of $E$ to $P$,

$$\deg\left(\mathrm{ad}(E_P)\right) \leq 0 .$$

**Point** $\mathrm{Bun}_{G_X}^{d,ss} \subset \mathrm{Bun}_{G_X}^{d}$ is an open substack and it admits a coarse moduli space $\mathcal{M}_{G_X}^{d,ss}$.

## Harder-Narasimhan type

**Theorem** (Atiyah-Bott, 1983)

Let $E \to X$ be a principal bundle. Then there exists a canonical reduction of structure group to a parabolic subgroup $P \subset G$ such that, if $L = P/R_u(P)$ is the Levi factor, the $L$-bundle $E_P \times_P L$ is semistable.

**Point** This induces a partition ("stratification") of $\mathrm{Bun}^d_{G_x}$ into $\underbrace{\text{Harder-Narasimhan types}}_{\substack{= \text{ topological type } \delta_\mu \text{ of } E_P \times_P L \\ \text{as an } L\text{-bundle}}}$.

# Perfect stratification

**Theorem** (Lin - S.)

For all $d \in \pi_1 G$,

$$H_{u,v}(\mathrm{Bun}^d_{G_x}; \mathbb{C}) = \sum_{\underbrace{\mu \in I(G,d)}_{\substack{\text{set of} \\ \text{Harder-Narasimhan} \\ \text{types for } G\text{-bundles} \\ \text{of degree } d}}} (uv)^{\overset{\substack{\text{codimension of} \\ \mathrm{Bun}_\mu \text{ in } \mathrm{Bun}^d_{G_x}}}{d_\mu}} H_{u,v}(\underbrace{\mathrm{Bun}_\mu}_{\substack{G\text{-bundles of} \\ \text{type } \mu}}; \mathbb{C})$$

**Point** $\forall j \geq 0$, only strata of codimension $\leq \frac{j}{2}$ contribute to the cohomology in degree $j$:

$$H^j(\mathrm{Bun}^d_{G_x}; \mathbb{C}) = H^j\Big(\bigcup_{\mu \mid d_\mu \leq \frac{j}{2}} \mathrm{Bun}_\mu; \mathbb{C}\Big)$$

## Recursive formula

The morphism of algebraic stacks

$$\begin{aligned} \mathrm{Bun}_\mu &\longrightarrow \mathrm{Bun}_{L_x}^{\delta_{\mu,ss}} \\ E &\longmapsto E_p \times_p L \end{aligned}$$

topological type $\delta_\mu \in \pi_1 L$ of $E_p \times_p L$ as an $L$-bundle

is an acyclic fibration, so

$$H_{u,v}(\mathrm{Bun}_\mu) \simeq H_{u,v}\left(\mathrm{Bun}_{L_x}^{\delta_{\mu,ss}}\right).$$

Therefore

$> 0$

$$HP_{u,v}\left(\mathrm{Bun}_{G_x}^{d,ss}\right) = H_{u,v}\left(\mathrm{Bun}_{G_x}^{d}\right) - \sum_{\mu \in I(G,d) \setminus \{\mu_{ss}\}} (uv)^{d_\mu} HP_{u,v}\left(\mathrm{Bun}_{L_x}^{\delta_{\mu,ss}}\right)$$

# The closed formula

generalizing the inversion formula of Laumon-Rapoport (1996) from one to two variables, we get:

Theorem (Lin-S.) Let $P \subset G$ be a parabolic subgroup and let $a_{u,v}(L(P)_x)$ be the Hodge series of $\mathrm{Bun}_{L(P)_x}^{\delta_\mu}$. There exists an (explicit) rational fraction $b_{u,v}(P) \in \mathbb{Q}(u,v)$, independent of $X$, such that

$$\boxed{HP_{u,v}\left(\mathrm{Bun}_{G_x}^{d,ss}\right) = \sum_{\substack{B \subset P \subset G \\ \text{parabolic}}} a_{u,v}(L(P)_x)\, b_{u,v}(P)\, (uv)^{(g_x - 1)\dim \underbrace{R_u(P)}_{\text{unipotent radical of } P}}}$$
.