NILPOTENT ORBITS AND THEIR FUNDAMENTAL GROUP IN THE CLASSICAL CASE

LIVA DILER

Abstract. As observed in the $\mathfrak{sl}_n$ case, nilpotent orbits are closely related to the set $\mathscr{P}(n)$ of partitions of n. This observation leads to the question if one can classify nilpotent orbits for other Lie algebras in the same fashion. We will handle the classical case, giving a complete classification of nilpotent G_{ad}-orbits in $\mathfrak{sl}_n, \mathfrak{sp}_{2n}, \mathfrak{so}_{2m+1}$ and $\mathfrak{so}_{2m}$. Moreover, we will show that this correspondence also behaves nicely when changing to a more interesting category than **Set**. Having studied the combinatorial nature of nilpotent orbits, we will apply the results from the first section to give a formula for the fundamental group $\pi_1(\mathscr{O}_X)$, as well as the G_{ad}-equivariant fundamental group $\mathscr{A}(\mathscr{O}_X)$ in the classical case. As an application, we will conclude by throwing a quick glance at the construction of explicit standard triples for $\mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$.

Contents

1. Preliminaries

Definition 1.1 (Partition)**.** *A* partition *of a natural number n is a tuple $[d_1, \ldots, d_n] \in \mathbb{N}^n$ such that*

$$\sum_i d_i = n \text{ and } d_1 \geq d_2 \geq \cdots \geq d_n$$

Two partitions $[d_1, \ldots, d_n]$ and $[p_1, \ldots, p_n]$ are said to be equal, if their nonzero parts agree. The set of all partitions of n is denoted $\mathscr{P}(n)$.

Remark 1.2.
- ⋆ $d_i \neq 0$ for all $i \iff d_i = 1$ for all i.
- ⋆ Occasionally, we will denote a partition $[d_1, \ldots, d_n]$ simply by $\mathbf{d}$.

Definition 1.3 (Exponential Notation). *We write* $[t_1^{i_1}, \dots, t_r^{i_r}]$ *to denote the partition* $[d_1, \dots, d_n]$*, where*

$$d_j = \begin{cases} t_1 & 1 \leq j \leq i_1 \\ t_2 & i_1 + 1 \leq j \leq i_1 + i_2 \\ t_3 & i_1 + 1_2 + 1 \leq j \leq i_1 + i_2 + i_3 \\ \vdots & \vdots \end{cases}$$

Example 1.4. In exponential notation, we write

$$[4, 3^2, 2^3, 1, 0^{10}] = [4, 3, 3, 2, 2, 2, 1, 0, \dots, 0]$$

for the partition of 17.

Definition 1.5 (Very even partition). *A partition* $[d_1, \dots, d_n]$ *of* n *is called* very even*, if for all* i*,* d_i *is even and has even multiplicity.*

2. Partition Type Classifications

Let $\epsilon = \pm 1$ and consider a non-degenerate form $\langle \cdot, \cdot \rangle_\epsilon$ on $\mathbb{C}^m$, such that

$$\langle A, B \rangle_\epsilon = \epsilon \langle B, A \rangle_\epsilon \text{ for all } A, B \in \mathbb{C}^m.$$

Remark 2.1.

- ⋆ If $\epsilon = -1$, $\langle \cdot, \cdot \rangle_\epsilon$ is symplectic.
- ⋆ If $\epsilon = 1$, $\langle \cdot, \cdot \rangle_\epsilon$ is symmetric.

Definition 2.2 (Isometry Group). *Denote by*

- ⋆ $I(\langle \cdot, \cdot \rangle_\epsilon) = \{x \in GL_m(\mathbb{C}) \mid \langle xA, xB \rangle_\epsilon = \langle A, B \rangle_\epsilon$ *for all* $A, B \in \mathbb{C}^m\}$ *the* isometry group *of* $\langle \cdot, \cdot \rangle_\epsilon$ *on* $\mathbb{C}^m$*, and by*
- ⋆ $\mathfrak{g}_\epsilon = \{X \in \mathfrak{sl}_m \mid \langle XA, B \rangle_\epsilon = -\langle A, XB \rangle_\epsilon$ *for all* $A, B \in \mathbb{C}^m\}$ *its Lie algebra.*

This definition is well defined: Since $I(\langle \cdot, \cdot \rangle_\epsilon)$ is a closed subgroup of the Lie group $GL_m(\mathbb{C})$, it is itself a Lie group by Cartans closed-subgroup theorem. Thus, one can speak of its Lie algebra.

Remark 2.3.

- ⋆ If $\epsilon = -1$, $m = 2n$ must be even, so $I(\langle \cdot, \cdot \rangle_\epsilon) = Sp_{2n}$.
- ⋆ If $\epsilon = 1$, $I(\langle \cdot, \cdot \rangle_\epsilon) \cong O_m$ and $\mathfrak{g}_1 \cong \mathfrak{so}_m$.

If $\epsilon = -1$, the adjoint group of $\mathfrak{g}_\epsilon$ is $PSp_{2n} := Sp_{2n}/\{\pm I\}$ and its orbits coincide with those of Sp_{2n}. If $\epsilon = 1$ and m is odd, then $I(\langle \cdot, \cdot \rangle_\epsilon) = O_m$ is the direct product its center $\{\pm I\}$ with the adjoint group SO_m of $\mathfrak{g}_\epsilon$, so again, the orbits coincide. The problem arises however, when $\epsilon = 1$ and m is even. Then the adjoint group of $\mathfrak{g}_\epsilon$ becomes $PSO_m := SO_m/\{\pm I\}$, and its orbits do not coincide with those of O_m. As we shall later see, there can only be one O_m-orbit attached to a very even partition $\mathbf{d} \in \mathscr{P}(m)$. It turns out that this orbit is the union $\mathscr{O}^I_{\mathbf{d}} \cup \mathscr{O}^{II}_{\mathbf{d}}$ of two orbits corresponding to $\mathbf{d}$.

Set

$$\mathscr{P}_\epsilon(m) = \{[d_1, \dots, d_n] \in \mathscr{P}(m) : \#\{j \mid d_j = i\} \text{ is even for all } i \text{ with } (-1)^i = \epsilon\}$$

Let $\mathfrak{g}$ be a classical Lie algebra with *standard representation* on $\mathbb{C}^n$, i.e.

$$X \cdot v := X(v) \text{ for all } X \in \mathfrak{g}, v \in \mathbb{C}^n$$

If $X \in \mathfrak{g}$ is nilpotent, then we can also regard X as a nilpotent element of $\mathfrak{sl}_n$. Then there is a corresponding partition $\mathbf{d} = [d_1, \dots, d_n]$ and moreover, belongs to a standard triple in $\mathfrak{sl}_n$. However, we can also attach to X a standard triple $\{H, X, Y\} \subset \mathfrak{g}$, which is conjugate under GL_n to the first triple. Denote by $\mathfrak{a}$ the span of $\{H, X, Y\}$.

Lemma 2.4. *The nonzero d_i are exactly the dimensions of the irreducible summands of the standard representation $\mathbb{C}^n$, regarded as an $\mathfrak{a}$-module.*

Our next goal is to establish a bijective correspondence between nilpotent orbits in $\mathfrak{sp}_{2n}$, resp. $\mathfrak{so}_m$, and certain partitions of $2n$, resp. m.
Let's start with the case $\mathfrak{g} = \mathfrak{sp}_{2n}$. Let $\langle\cdot,\cdot\rangle$ be the non-degenerate symplectic form on $\mathbb{C}^{2n}$ which is preserved by G_{ad}. We get an $\mathfrak{a}$-module decomposition

$$\mathbb{C}^{2n} = \bigoplus_{r\geq 0} M(r)$$

where $M(r)$ is a finite direct sum of irreducible $\mathfrak{a}$-modules (i.e. representations of $\mathfrak{sl}_2$) of highest weight r. By the above Lemma, we can read off the dimension of the summands from the partition $[d_1, \dots, d_n]$ of X, regarded as a matrix in $\mathfrak{sl}_{2n}$. For $r \geq 0$, denote by $H(r)$ the highest weight space in $M(r)$. Note that

$$\dim H(r) = \mathrm{mult}(\rho_r, M(r))$$

where ρ_r denotes the irreducible $\mathfrak{a}$-module of highest weight. Now, to equip $H(r)$ with a bilinear form, put

$$(v, w)_r := \langle v, Y^r \cdot w\rangle \text{ for all } v, w \in H(r)$$

Lemma 2.5. *The form $(\cdot,\cdot)_r$ is symplectic (resp. symmetric) if r is even (resp. odd).*

Proof. Using $\mathfrak{g}$-invariance, we get

$$\begin{aligned}
(v, w)_r &= \langle v, Y^r \cdot w\rangle \\
&= \langle v, \mathrm{ad}_Y^r(w)\rangle \\
&= \langle [v, Y] \cdot Y^{r-1}, w\rangle \\
&= \langle [\dots[v, Y]\dots, Y], w\rangle \\
&= \begin{cases} \langle Y^r \cdot v, w\rangle & r \text{ even} \\ -\langle Y^r \cdot v, w\rangle & r \text{ odd} \end{cases} \\
&= \begin{cases} -(w, v) & r \text{ odd} \\ (w, v) & r \text{ even} \end{cases}
\end{aligned}$$

□

Lemma 2.6. *The form $(\cdot,\cdot)_r$ is non-degenerate for all r.*

Proof. Note that the r-weight space of $\mathbb{C}^{2n}$ is $\langle\cdot,\cdot\rangle$-orthogonal to its s-weight space, whenever $s \neq -r$, by the invariance of ad_H relative to $\langle\cdot,\cdot\rangle$. Suppose $r \geq 0$. Then $H(r)$ has a canonical complement in the full r-weight space. It is spanned by all vectors in this weight space lying in $\langle Y\rangle$. Since $Y^{r+1} \cdot H(r) = 0$, we see that $H(r)$ is orthogonal to this complement with respect to $(\cdot,\cdot)_r$. By $\mathfrak{sl}_2$ theory, $Y^r \cdot H(r)$ is the lowest weight space in $M(r)$, and it pairs non-degenerately with $H(r)$ via $\langle\cdot,\cdot\rangle$. Thus, $(\cdot,\cdot)_r$ is non-degenerate. □

Since the irreducible representation of highest weight r has dimension $r+1$ and non-degenerate symplectic forms exist only in even dimension, we deduce the following result.

Corollary 2.7. *The partition $[d_1, \dots, d_n]$ of X lies in $\mathscr{P}_{-1}(2n)$, i.e. its odd parts occur with even multiplicity.*

Thus, we get a well-defined map

$$\begin{aligned}\Pi_{-1} : \{\text{nilpotent } I(\langle\cdot,\cdot\rangle)\text{-orbits in } \mathfrak{sp}_{2n}\} &\to \mathscr{P}_{-1}(2n)\\ \mathscr{O}_{X_{[d_1,\dots,d_{2n}]}} &\mapsto [d_1, \dots, d_{2n}]\end{aligned}$$

The case $\mathfrak{g} = \mathfrak{so}_m$ is analogous. Again, let $\langle\cdot,\cdot\rangle$ be the non-degenerate form on $\mathbb{C}^m$ preserved by G_{ad}. Denote by $\mathfrak{a}$ the span of a standard triple $\{H, X, Y\}$. Consider again the decomposition

$$\mathbb{C}^m = \bigoplus_{r\geq 0} M(r)$$

and define $H(r)$ and $(\cdot,\cdot)_r$ exactly as above.

Lemma 2.8. *The form $(\cdot,\cdot)_r$ is symmetric (resp. symplectic) if r is even (resp. odd).*

Corollary 2.9. *The partition $[d_1, \dots, d_n]$ of X lies in $\mathscr{P}_1(m)$, i.e. its odd parts occur with even multiplicity.*

Thus, we get a well-defined map

$$\begin{aligned}\Pi_{1} : \{\text{nilpotent } I(\langle\cdot,\cdot\rangle)\text{-orbits in } \mathfrak{so}_{m}\} &\to \mathscr{P}_{1}(m)\\ \mathscr{O}_{X_{[d_1,\dots,d_{m}]}} &\mapsto [d_1, \dots, d_{m}]\end{aligned}$$

Lemma 2.10 (Wall)**.** *The maps $\Pi_{\pm 1}$ are bijections.*

Proof. We will treat the case $\mathfrak{g} = \mathfrak{sp}_{2m}$, the case $\mathfrak{g} = \mathfrak{so}_m$ is similar. To prove surjectivity, let $\mathbf{d} = [d_1^{i_1}, \dots, d_r^{i_r}] \in \mathscr{P}_{-1}(2n)$ and define a vector space

$$V = \bigoplus_{j=1}^{r} V_j$$

where $\dim V_j = i_j$. We want to define a form $(\cdot,\cdot) : V \times V \to \mathbb{C}$ on V as follows: $(V_i, V_j) = 0$ if $i \neq j$. Moreover, if d_j is odd (resp. even), we require $(\cdot,\cdot)|_{V_j\times V_j}$ to be non-degenerate and symplectic (resp. symmetric). Note that such a form exists, and is unique up to equivalence. Now for $d_j \neq 1$, replace the summands V_j by $W_j \oplus W_j'$, where W_j, W_j' are isomorphic copies of V_j. Now V is a subspace of the larger vector space

$$\bigoplus_{j=1,d_j\neq 1}^{r} W_j \oplus W_j' \oplus \bigoplus_{j=1,d_j=1}^{r} V_j$$

For $d_j \neq 1$, replace $(\cdot,\cdot)$ on V_j by a symplectic form $\langle\cdot,\cdot\rangle_j$ on $W_j \oplus W_j'$ such that W_j is paired non-degenerately with W_j' and each of W_j and W_j' is self orthogonal. Again up to equivalence, there is a unique way to do this. Consider now a symplectic form $\langle\cdot,\cdot\rangle'$ on $W = \bigoplus_j W_j \oplus W_j'$, which is just the orthogonal sum of the $\langle\cdot,\cdot\rangle_j$. Using the formulas in Lemma 7.2.1 in [Hum72] for the action of the standard basis vectors of $\mathfrak{sl}_2$ on a finite-dimensional irreducible module, we enlarge each $W_j \oplus W_j'$ to a $d_j i_j$-dimensional $\mathfrak{sl}_2$-module, whose highest weight space is W_j and whose lowest weight space is W_j'. This module is the direct sum of i_j irreducible submodules, each of highest weight $d_j - 1$. It admits a non-degenerate symplectic form extending $\langle\cdot,\cdot\rangle_j$

and invariant under the $\mathfrak{sl}_2$-action. By Schur's Lemma, this form is unique up to $\mathfrak{sl}_2$-equivariant equivalence. If $d_k = 1$, then V_k may be regarded as a trivial $\mathfrak{sl}_2$-module with a non-degenerate symplectic form $\langle \cdot, \cdot \rangle$. Now, denote by V' the direct sum of all these $\mathfrak{sl}_2$-modules with the inherited symplectic form. Then V' is isomorphic to the standard representation $\mathbb{C}^{2n}$. Clearly, $\mathfrak{sp}(V)$ has a nilpotent element with partition $\mathbf{d}$. Hence, Π_{-1} is surjective. For injectivity, note that any two images of $\mathfrak{sl}_2$ in $\mathfrak{sp}_{2n}$ giving rise to the same partition of $2n$ must be conjugate under an isometry of the symplectic form. □

Thus we get the following classification results.

Theorem 2.11 (Type B_N)**.** *There is a* 1 : 1*-correspondence*

$$\{\textit{Nilpotent orbits in } \mathfrak{so}_{2n+1}\} \longleftrightarrow \mathscr{P}_1(2n+1)$$

Theorem 2.12 (Type C_N)**.** *There is a* 1 : 1*-correspondence*

$$\{\textit{Nilpotent orbits in } \mathfrak{sp}_{2n}\} \longleftrightarrow \mathscr{P}_{-1}(2n+1)$$

Theorem 2.13 (Gerstenhaber)**.** *There is a* 1 : 1*-correspondence*

$$\{\textit{Nilpotent } I(\langle \cdot, \cdot \rangle)\textit{-orbits in } \mathfrak{g}_\epsilon\} \longleftrightarrow \mathscr{P}_\epsilon(m)$$

Example 2.14.

- ⋆ In $\mathfrak{so}_7$, there are seven nilpotent orbits, namely

$$\mathscr{O}_{[7]}, \mathscr{O}_{[5,1^2]}, \mathscr{O}_{[3,1^4]}, \mathscr{O}_{[3,2^2]}, \mathscr{O}_{[3^2,1]}, \mathscr{O}_{[2^3,1^3]}, \mathscr{O}_{[1^7]}$$

- ⋆ In $\mathfrak{sp}_6$, there are eight nilpotent orbits, namely

$$\mathscr{O}_{[6]}, \mathscr{O}_{[4,2]}, \mathscr{O}_{[4,1^2]}, \mathscr{O}_{[3^2]}, \mathscr{O}_{[2^3]}, \mathscr{O}_{[2^2,1^2]}, \mathscr{O}_{[2,1^4]}, \mathscr{O}_{[1^6]}$$

However, we are not quite satisified yet; what about nilpotent orbits in $\mathfrak{so}_{2n}$? We shall classify them now.

Theorem 2.15 (Type D_n, Springer-Steinberg)**.** *Nilpotent orbits in $\mathfrak{so}_{2n}$ are parametrized by partitions of* $2n$ *in which even parts occur with even multiplicity, except that very even partitions* $\mathbf{d}$ *correspond to two orbits, denoted* $\mathscr{O}^I_{\mathbf{d}}$ *and* $O^{II}_{\mathbf{d}}$.

The reason we can't prove this in the same fashion as for Type B_n and C_n, is that for $\mathfrak{g} = \mathfrak{so}_m$ the adjoint group G_{ad} is isomorphic to PSO_m, and while the PSO_m-orbits coincide with the SO_m-orbits, they do not coincide with the $I(\langle \cdot, \cdot \rangle) \cong O_m$-orbits if m is even.

Proof of Theorem 2.15. Let $m = 2n$ and, given two actions of $\mathfrak{sl}_2$ on $\mathbb{C}^m$ invariant under $\langle \cdot, \cdot \rangle_1$, suppose they are conjugate under an element of $g \in I(\langle \cdot, \cdot \rangle_1)$. Suppose that the determinant of the matrix g is -1; then we must decide when we can replace g by a matrix of determinant 1. Assume first, that at least one part of the partition $\mathbf{d}$ corresponding to either action of $\mathfrak{sl}_2$ is odd. Then the proof of 2.10 shows that we can find an irreducible odd-dimensional summand of $\mathbb{C}^m$ under the first action that pairs non-degenerately with itself under $\langle \cdot, \cdot \rangle_1$. Multiplying g by -1 on this summand S and leaving it unchanged on the orthogonal complement of S, we obtain a new g that also conjugates the first action to second but has determinant 1. Hence, the two actions are already conjugate under SO_m or PSO_m. Now assume that all parts of $\mathbf{d}$ are even, so they all occur with even multiplicity. Then again, the proof of 2.10 shows that the commutant in O_m of either $\mathfrak{sl}_2$-action is the direct product of symplectic groups, one for each distinct part of $\mathbf{d}$. Since a symplectic transformation automatically has determinant 1, it is impossible to replace g by any g of determinant 1. Hence, very even partitions of m correspond to two orbits: Given a representative of one of them, one obtains a representative of the other by conjugating by an orthogonal matrix of determinant −1. Other partitions of m correspond to one orbit. □

3. Topology of Nilpotent Orbits

3.1. **The Closure Ordering.** Recall the partial ordering on the set of nilpotent orbits, given by the Zariski closure operation: For a nilpotent element $X \in \mathfrak{g}$, we set

$$\mathcal{O}_X \leq \mathcal{O}_{X'} :\Longleftrightarrow \overline{\mathcal{O}_X} \subset \overline{\mathcal{O}_{X'}}$$

where $\overline{\mathcal{O}_X}$ is the Zariski-closure of $\mathcal{O}_X$. In this section, we want to build a bridge to the previous partition-type classifications of nilpotent orbits in the classical Lie algebras.

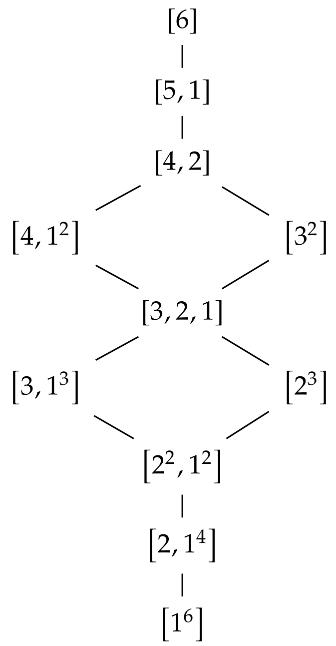
Definition 3.1 (Partial order on $\mathscr{P}(N)$)**.** *Given $\boldsymbol{f} = [f_1, \ldots, f_N], \boldsymbol{d} = [d_1, \ldots, d_N] \in \mathscr{P}(N)$, we say that $\boldsymbol{d}$* dominates *$\boldsymbol{f}$, denoted by $\boldsymbol{d} \geq \boldsymbol{f}$, if*

$$\sum_{1 \leq j \leq k} d_j \geq \sum_{1 \leq j \leq k} f_j \text{ for all } k \leq N$$

We say that $\boldsymbol{d}$ covers *$\boldsymbol{f}$, if $\boldsymbol{d} > \boldsymbol{f}$ and there is no partition $\boldsymbol{e}$ such that $\boldsymbol{d} > \boldsymbol{e} > \boldsymbol{f}$.*

This partial order is usually referd to as the *dominance order*.

Example 3.2. Let $N = 6$. We can visualize $(\mathscr{P}(6), \geq)$ as follows:



Lemma 3.3. *Let $\mathcal{O}_{\boldsymbol{d}}$ and $\mathcal{O}_{\boldsymbol{f}}$ be nilpotent orbits in $\mathfrak{sl}_n$ corresponding to $\boldsymbol{d}$ and $\boldsymbol{f}$ and let $X \in \mathcal{O}_{\boldsymbol{d}}, Y \in \mathcal{O}_{\boldsymbol{f}}$. Then $\boldsymbol{d} \geq \boldsymbol{f}$ if and only if* $\operatorname{rank}(X^k) \geq \operatorname{rank}(Y^k)$ *for all $k \geq 0$.*

Proof. It can be computed, that

$$\operatorname{rank}(X^k) = \sum_{\{i \mid d_i \geq k\}} (d_i - k)$$

Suppose that $\mathbf{d} \not\geq \mathbf{f}$ and let j be the smallest integer with

$$\sum_{i=1}^{j} d_i < \sum_{i=1}^{j} f_i.$$

Clearly, $d_j < f_j$. No term d_i with $i > j$ contributes to $\mathrm{rank}(X^{d_j})$, so $\mathrm{rank}(X^{d_j}) < \mathrm{rank}(Y^{d_j})$. Conversely, suppose that $\mathrm{rank}(X^k) < \mathrm{rank}(Y^k)$ for some k and let m be the largest index with $f_m \geq k$. Then

$$\mathrm{rank}(Y^k) = \sum_{i=1}^{m}(f_i - k),$$

while

$$\sum_{i=1}^{m}(d_i - k) \leq \mathrm{rank}(X^k).$$

Hence

$$\sum_{i=1}^{m} d_i < \sum_{i=1}^{m} f_i,$$

so that $\mathbf{d} \not\geq \mathbf{f}$. □

Lemma 3.4 (Gerstenhaber)**.** *Let $\boldsymbol{d}, \boldsymbol{f} \in \mathscr{P}(N)$ with $\boldsymbol{d} = [d_1, \dots, d_N]$. Then $\boldsymbol{d}$ covers $\boldsymbol{f}$ if and only if $\boldsymbol{f}$ can be obtained from $\boldsymbol{d}$ by the following procedure: Choose an index i and let j be the smallest index greater than i such that $0 \leq d_j < d_i - 1$. Assume that either $d_j = d_i - 2$ or $d_k = d_i$ whenever $i < k < j$. Then the parts of $\boldsymbol{f}$ are obtained by from the d_k by replacing d_i, d_j by $d_i - 1, d_j + 1$.*

Proof. See Lemma 6.2.4 in [CM93]. □

Theorem 3.5 (Gerstenhaber, Hesselink)**.** *Let $\mathfrak{g}$ be a classical Lie algebra, and let $\boldsymbol{d}, \boldsymbol{f}$ be partitions of two nilpotent orbits $\mathcal{O}_{\boldsymbol{d}}, \mathcal{O}_{\boldsymbol{f}}$ in $\mathfrak{g}$. Then $\mathcal{O}_{\boldsymbol{d}} > \mathcal{O}_{\boldsymbol{f}}$ if and only if $\boldsymbol{d} > \boldsymbol{f}$.*

Proof. Let $X \in \mathcal{O}_{\mathbf{d}}, Y \in \mathcal{O}_{\mathbf{f}}$. Since the rank of any power of a matrix is invariant under conjugation, and since the condition that the rank of a matrix is a zariski-closed condition (because $\mathrm{cod}(\mathrm{rank}(\text{-})) = \mathbb{N}$, i.e. discrete), we can deduce

$$\mathcal{O}_{\mathbf{d}} > \mathcal{O}_{\mathbf{f}} \implies \mathrm{rank}(X^k) > \mathrm{rank}(Y^k) \text{ for all } k$$
$$\overset{3.3}{\iff} \mathbf{d} > \mathbf{f}$$

We will prove the converse for $\mathfrak{g} = \mathfrak{sl}_n$ case and refer the reader to [Hes76] for the more general case. Let $\mathbf{d} > \mathbf{f}$ and assume that without loss of generality, $\mathbf{d}$ covers $\mathbf{f}$. Chose a standard triple in $\mathfrak{g}$ with $X \in \mathcal{O}_{\mathbf{d}}$ as in 1 and define the subalgebra

$$\mathfrak{q}_2 = \sum_{i \geq 2} \mathfrak{g}_i,$$

where

$$\mathfrak{g}_i = \{Z \in \mathfrak{g} \mid \mathrm{ad}_H Z = [H, Z] = iZ\}$$

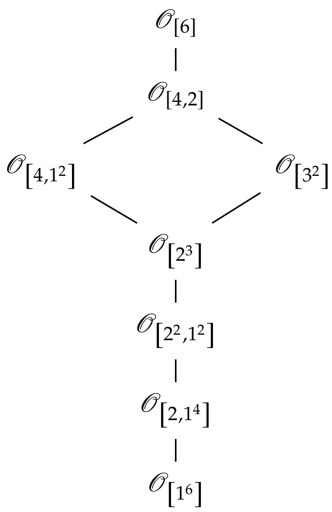
Using 1, we can see that $\mathcal{O}_{\mathbf{f}}$ is represented by an element of $\mathfrak{q}^2$. By a Lemma of Kostant (Lemma 4.1.4 in [CM93]), the desired result follows. □

Note that we wrote $>$ instead of $\geq$ since for Type D, we have two orbits attached to a very even partition which are incomparable because they have the same dimension. But we still get:

Corollary 3.6. *Let $\mathfrak{g}$ be a Lie algebra of A, B or C. Let $\boldsymbol{d}, \boldsymbol{f}$ be partitions of two nilpotent orbits $\mathcal{O}_{\boldsymbol{d}}, \mathcal{O}_{\boldsymbol{f}}$ in $\mathfrak{g}$. Then $\mathcal{O}_{\boldsymbol{d}} \geq \mathcal{O}_{\boldsymbol{f}}$, if and only if $\boldsymbol{d} \geq \boldsymbol{f}$.*

This tells us that the bijections established in 2.11 can be regarded as an isomorphism in a slightly more interesting category than **Set**, namely the category of posets. Moreover, $(\mathcal{N}, \geq)$ and $(\mathscr{P}_{\mathfrak{g}}(N), \geq)$[1] are equivalent, regarded as poset category.

Example 3.7.
(1) Let $\mathfrak{g} = \mathfrak{sl}_6$. Then the diagram of nilpotent orbits in coincides with the diagram given above.
(2) Let $\mathfrak{g} = \mathfrak{sp}_6$. We can visualize $(\mathcal{N}, \geq)$ as follows:



For more diagrams in the classical, as well as the exceptional case, see Chapter 4 in [Spa82].

3.2. **The Fundamental Group and $\mathscr{A}(\mathcal{O})$.** The goal of this section is to study the fundamental group of a given nilpotent orbit $\mathcal{O}_X$ in $\mathfrak{g}$. It turns out that its useful to study the universal cover $\tilde{\mathcal{O}}_X$ of $\mathcal{O}_X$. Recall that the universal covering $p : G_{\text{sc}} \longrightarrow G_{\text{ad}}$ has a natural complex Lie group structure (c.f. Prop. 7.9 in [FH91]). In particular, p is a homomorphism of Lie groups whose kernel is precisely the center Z of G_{sc}. Recall the following definition:

Definition 3.8 (Homogeneous Space)**.** *Let $\mathscr{C}$ be a locally small category which admits a functor $U : \mathscr{C} \longrightarrow$ **Set**, X an object of $\mathscr{C}$ and G a group. Given a group homomorphism*

$$\eta : G \longrightarrow \text{Aut}_{\mathscr{C}}(X),$$
$$g \longmapsto \eta_g$$

the triple (X, η, U) is called a homogeneous space *for G, if G acts transitively, i.e. the map*

$$G \times U(X) \longrightarrow U(X) \times U(X)$$
$$(g, x) \longmapsto (x, \eta_g(x))$$

is surjective.

[1] $\mathscr{P}_{\mathfrak{g}}(N)$ denotes the set of partitions corresponding to $\mathfrak{g}$ via 2.11.

Before computing the fundamental group of $\mathcal{O}_X$, we shall explain how to get an action of G_{sc} on $\tilde{\mathcal{O}}_X$: Recall that for a path-connected, locally path-connected, locally relatively simply connected pointed space (X, x_0), the (up to isomorphism) unique simply connected covering space is given by

$$\tilde{X} = \{[f] \operatorname{rel} \partial I \mid f \text{ is a path in } X \text{ with } f(0) = x_0\}$$

topologized in the usual fashion (c.f. Thm. 8.4 in [Bre93]). Now let

$$\begin{aligned} G \times X &\to X, \\ (g, x) &\mapsto g \cdot x \end{aligned}$$

be an action of a Lie group on a space X. Since the universal covering $p : \tilde{G} \to G$ is a surjective homomorphism, composition yields a lift of the action of G to an action of $\tilde{G}$

$$\begin{array}{ccc} \tilde{G} \times X & & \\ \downarrow & \searrow & \\ G \times X & \longrightarrow & X \end{array}$$

We are now in the position to lift the action of $\tilde{G}$ on X to an action on $\tilde{X}$. We define

$$\tilde{G} \times \tilde{X} \to \tilde{X}, (g, \gamma) \mapsto (\omega : t \mapsto g(t) \cdot \gamma(t))$$

and get a well-defined group action. Obviously, the following diagram commutes:

$$\begin{array}{ccc} \tilde{G} \times \tilde{X} & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ G \times X & \longrightarrow & X \end{array}$$

We will now return to the usual setting where $\mathfrak{g}$ is a classical Lie algebra and $\mathcal{O}_X$ a nilpotent orbit in $\mathfrak{g}$.

Lemma 3.9. *(1) $\tilde{\mathcal{O}}_X \cong G_{sc}/(G_{sc}^X)^\circ$. Moreover, $\tilde{\mathcal{O}}_X$ is a homogeneous G_{sc}-space.*
(2) The group $\pi_1(\mathcal{O}_X)$ is isomorphic to the component group $G_{sc}^X/(G_{sc}^X)^\circ$ of the centralizer of X in G_{sc}.

Proof. (1) By simple connectedness of G_{sc}, the action is transitive, proving the first claim. Let $X' \in F := p^{-1}(\{X\})$ where $p : \tilde{\mathcal{O}}_X \to \mathcal{O}_X$ is the covering map. Consider an element $Y \in (G_{\mathrm{sc}}^X)^\circ$. Then

$$\begin{aligned} p(Y \cdot X') &= Y \cdot p(X') \\ &= Y \cdot X \\ &= X \end{aligned}$$

Thus, the $(G_{\mathrm{sc}}^X)^\circ$-Orbit of X' is a connected subspace of F, hence equal to $\{X'\}$ by discreteness of the fiber. We have $(G_{\mathrm{sc}}^X)^\circ \subset \mathrm{stab}_{X'}(G_{\mathrm{sc}})$ and get a covering

$$G_{\mathrm{sc}}/(G_{\mathrm{sc}}^X)^\circ \to \tilde{\mathcal{O}}_X$$

On the other hand, we have a covering

$$\begin{array}{c} G_{\mathrm{sc}}/(G_{\mathrm{sc}}^X)^\circ \\ \big\downarrow \\ \mathscr{O}_X \end{array}$$

which must in turn be covered by $\tilde{\mathscr{O}}_X$, yielding an isomorphism of coverings

$$\begin{array}{ccc} \tilde{\mathscr{O}}_X & \xrightarrow{\sim} & G_{\mathrm{sc}}/(G_{\mathrm{sc}}^X)^\circ \\ & \searrow \quad \swarrow & \\ & \mathscr{O}_X & \end{array}$$

(2) Since $(G_{\mathrm{sc}}^X)^\circ$ acts trivially on the fiber, we get

$$\begin{aligned} \mathrm{Deck}(\tilde{\mathscr{O}}_X) &\overset{(1)}{=} \mathrm{Deck}(G_{\mathrm{sc}}/(G_{\mathrm{sc}}^X)^\circ) \\ &= G_{\mathrm{sc}}^X/(G_{\mathrm{sc}}^X)^\circ. \end{aligned}$$

Thus,

$$\begin{aligned} \pi_1(\mathscr{O}_X) &= \mathrm{Deck}(\tilde{\mathscr{O}}_X) \\ &= G_{\mathrm{sc}}^X/(G_{\mathrm{sc}}^X)^\circ \end{aligned}$$

□

Definition 3.10 (G-equivariant Fundamental Group)**.** *Let G be a complex Lie group with Lie algebra $\mathfrak{g}$ and $\mathscr{O}_X$ a nilpotent orbit. The group*

$$\pi_1^G(\mathscr{O}_X) := G^X/(G^X)^\circ$$

is called the G-equivariant fundament group *of $\mathscr{O}_X$.*

Note that $\pi_1^G(\mathscr{O}_X)$ is the Deck transformation group of the largest covering space with a G-action. By 3.9, we have

$$\pi_1^{G_{\mathrm{sc}}}(\mathscr{O}_X) = G_{\mathrm{sc}}^X/(G_{\mathrm{sc}}^X)^\circ \cong \pi_1(\mathscr{O}_X)$$

We write $\mathscr{A}(\mathscr{O}_X) = \pi_1^{G_{\mathrm{ad}}}(\mathscr{O}_X)$. Recall that, given a nilpotent element $X \in \mathfrak{g}$, we can construct a standard triple $\{H, X, Y\}$ using Jacobson-Morozov and get a unique homomorphism

$$\phi : \mathfrak{sl}_2 \longrightarrow \mathfrak{g}$$

which is determined by the standard triple. We set

$$\mathfrak{g}^\phi := \{Z \in \mathfrak{g} \mid [Z, V] = 0 \text{ for all } V \in \mathfrak{a}\}$$

where $\mathfrak{a} = \mathbb{C}\langle H, X, Y\rangle$. Similarly, let G_{ad}^ϕ denote the centralizer of $\mathfrak{a}$ in G_{ad}. By 3.7.5 in [CM93], we have

$$G^X/(G^X)^\circ = G^\phi/(G^\phi)^\circ$$

Thus, we are reduced to studying the centralizier of $\mathrm{im}(\phi)$ in G. Assume now, that $\mathfrak{g}$ is classical.

Example 3.11.
- ⋆ If $\mathfrak{g} = \mathfrak{sl}_n$, then $G_{\mathrm{sc}} = SL_n$.
- ⋆ If $\mathfrak{g} = \mathfrak{sp}_{2n}$, then $G_{\mathrm{sc}} = Sp_{2n}$.
- ⋆ If $\mathfrak{g} = \mathfrak{so}_N$, then G_{sc} is a double cover for SO_N, denoted $Spin_N$.

Notation.

- ⋆ If H is any group, let $H^n_\Delta = \iota(H)$ denote the diagonal copy of H inside $\prod_{i=1}^n H$.
- ⋆ If $H_1, \dots, H_n$ are matrix groups, let $S(\prod_{i=1}^n H_i)$ denote the subgroup of $\prod_{i=1}^n H_i$ consisting of m-tuples of matrices with determinant 1.

Remark 3.12. $S(H\times K\times\dots)$ is not necessarily isomorphic to $S(H^n_\Delta\times K^m_\Delta\times\dots)$, although $H^n_\Delta \cong H$. For example if $H = GL_n$, we have $S(H) = SL_n$ but

$$S(H^2_\Delta) = SL_n^\pm = \{X \in GL_n \mid \det(X) = \pm 1\}.$$

Theorem 3.13 (Springer-Steinberg)**.** *Let $\mathfrak{g}$ be a classical Lie algebra and $\mathcal{O}_X$ a nilpotent orbit in $\mathfrak{g}$. Write $\mathcal{O}_X = \mathcal{O}_{[d_1,\dots,d_N]}$ for some $\mathbf{d} = [d_1, \dots, d_N] \in \mathscr{P}(N)$. Let $r_i = \#\{j \mid d_j = i\}$ be the multiplicities and $s_i = \#\{j \mid d_j \geq i\}$. Then*

$$G^\phi_{sc} \cong \begin{cases} S(\prod_i (GL_{r_i})^i_\Delta) & \mathfrak{g} = \mathfrak{sl}_n \\ \prod_{i\ odd}(Sp_{r_i})^i_\Delta \times \prod_{i\ even}(O_{r_i})^i_\Delta & \mathfrak{g} = \mathfrak{sp}_{2n} \\ \text{double cover of } C := S(\prod_{i\ even}(Sp_{r_i})^i_\Delta \times \prod_{i\ odd}(O_{r_i})^i_\Delta) & \mathfrak{g} = \mathfrak{so}_N \end{cases}$$

$$G^\phi_{\mathrm{ad}} \cong \begin{cases} S(\prod_i (GL_{r_i})^i_\Delta)/\{\text{scalar matrices in } SL_n\} & \mathfrak{g} = \mathfrak{sl}_n \\ G^\phi_{sc}/\{\pm I\} & \mathfrak{g} = \mathfrak{sp}_{2n} \\ C & \mathfrak{g} = \mathfrak{so}_{2n+1} \\ C/\{\pm I\} & \mathfrak{g} = \mathfrak{so}_{2n}. \end{cases}$$

In addition, the dimension of $\mathfrak{g}^X$ is given by

$$\dim(\mathfrak{g}^X) = \begin{cases} \sum_i s_i^2 - 1 & \mathfrak{g} = \mathfrak{sl}_n \\ \frac{1}{2}\sum_i s_i^2 + \frac{1}{2}\sum_{i\ odd} r_i & \mathfrak{g} = \mathfrak{sp}_{2n} \\ \frac{1}{2}\sum_i s_i^2 - \frac{1}{2}\sum_{i\ odd} r_i & \mathfrak{g} = \mathfrak{so}_N. \end{cases}$$

Proof. Theorem 6.1.3 in [CM93]. □

The dimension formula

$$\dim(\mathcal{O}_X) = \dim(\mathfrak{g}) - \dim(\mathfrak{g}^X)$$

from 1.2.15 in [CM93] yields

Corollary 3.14.

$$\dim(\mathcal{O}_X) = \begin{cases} n^2 - \sum_i s_i^2 & \mathfrak{g} = \mathfrak{sl}_n \\ 2n^2 + n - \frac{1}{2}\sum_i s_i^2 + \frac{1}{2}\sum_{i\ odd} r_i & \mathfrak{g} = \mathfrak{so}_{2n+1} \\ 2n^2 + n - \frac{1}{2}\sum_i s_i^2 - \frac{1}{2}\sum_{i\ odd} r_i & \mathfrak{g} = \mathfrak{sp}_{2n} \\ 2n^2 - n - \frac{1}{2}\sum_i s_i^2 + \frac{1}{2}\sum_{i\ odd} r_i & \mathfrak{g} = \mathfrak{so}_{2n} \end{cases}$$

Example 3.15.

(1) Let $\mathfrak{g} = \mathfrak{sl}_6$ and $\mathcal{O} = \mathcal{O}_{[2^3]}$. Then

$$r_1 = 0 \qquad s_1 = 3$$
$$r_2 = 3 \qquad s_2 = 3$$
$$r_3 = 0 \qquad s_3 = 0$$

so by 3.13, we have

$$G^\phi_{\mathrm{sc}} \cong S((GL_3)^2_\Delta) \cong SL_3^\pm$$

This group has two connected components, though $G^{\phi}_{\text{ad}} \cong SL_3$ is connected. It follows that

$$\begin{aligned}\pi_1(\mathcal{O}) &\overset{3.9}{\cong} G^X_{\text{sc}}/(G^X_{\text{sc}})^\circ \\ &\cong G^{\phi}_{\text{sc}}/(G^{\phi}_{\text{sc}})^\circ \\ &\cong SL^{\pm}_3/SL_3 \\ &\cong \mathbb{Z}/2\mathbb{Z}\end{aligned}$$

and

$$\mathscr{A}(\mathcal{O}) \cong \{1\}$$

Let $X \in \mathcal{O}$, then the dimension of $\mathcal{O}$ is given by

$$\begin{aligned}\dim(\mathcal{O}) &= \dim(\mathfrak{g}) - \dim(\mathfrak{g}^X) \\ &\overset{3.13}{=} 35 - 17 \\ &= 18\end{aligned}$$

(2) Let $\mathfrak{g} = \mathfrak{sl}_{10}$, $\mathcal{O} = \mathcal{O}_{[7,3]}$. Then $r_3 = r_7 = 1$ and $r_i = 0$ for all $i \neq 3, 7$. Now

$$G^{\phi}_{\text{sc}} \cong S(GL_1 \times GL_1) \cong GL_1 \cong \mathbb{C}^\times \cong G^{\phi}_{\text{ad}},$$

which is connected. Thus

$$\pi_1(\mathcal{O}) = \mathscr{A}(\mathcal{O}) = \{1\}$$

(3) Let $\mathfrak{g} = \mathfrak{sp}_{12}$ and consider the orbit $\mathcal{O} = \mathcal{O}_{[4^2,2^2]}$. Now $r_2 = r_4 = 2$ while $r_1 = r_3 = 0$. We have

$$G^{\phi}_{\text{sc}} \cong (O_2)^4_\Delta \times (O_2)^2_\Delta$$

and

$$G^{\phi}_{\text{ad}} \cong G^{\phi}_{\text{sc}}/\{\pm I\},$$

so

$$\pi_1(\mathcal{O}) \cong \mathscr{A}(\mathcal{O}) \cong (\mathbb{Z}/2\mathbb{Z})^2$$

For $X \in \mathcal{O}$, we have $\dim(\mathfrak{g}^X) = 20$, hence $\dim(\mathcal{O}) = 58$.

(4) Let $\mathfrak{g} = \mathfrak{so}_{12}$, $\mathcal{O} = \mathcal{O}_{3^2,2^2,1^2}$. Then 3.13 tells us, that G^{ϕ}_{sc} is a double cover of $S((O_2)^3_\Delta \times (Sp_2)^2_\Delta \times O_2)$ which can also be regarded as an index 2 subgroup of $Pin_2 \times Sp_2 \times O_2$, where Pin_n is a double cover of O_n corresponding to the double cover $Spin_n$ of SO_n. We have $G^{\phi}_{\text{ad}} = G^{\phi}_{\text{sc}}/\{\pm I\}$ and

$$\pi_1(\mathcal{O}) = \mathscr{A}(\mathcal{O}) = \mathbb{Z}/2\mathbb{Z}.$$

Next, we want give formulae for $\pi_1(\mathcal{O})$ and $\mathscr{A}(\mathcal{O})$ of any nilpotent orbit $\mathcal{O} = \mathcal{O}_{[d_1,\dots,d_N]}$ in a classical Lie algebra $\mathfrak{g}$.

Notation. We set

$$\begin{aligned}a &= \text{ number of distinct odd } d_i \\ b &= \text{ number of distinct even nonzero } d_i \\ c &= \gcd(d_1, \dots, d_N)\end{aligned}$$

Definition 3.16.

- ⋆ *A group E is called a* central extension *of a group H by a group K, if there exists a short exact sequence*
$$1 \to K \to E \to H \to 1$$
such that K is a central subgroup of E.
- ⋆ *A partition is called* rather odd, *if all of its odd parts have multiplicity one.*

Remark 3.17. $\pi_1(G_{\mathrm{ad}}, 1)$ lies in the center of G_{sc}, and the sequence
$$1 \to \pi_1(G_{\mathrm{ad}}, 1) \to G_{\mathrm{sc}} \to G \to 1$$
is exact. Consequently, G_{sc} is a central extension of G_{ad} by $\pi_1(G_{\mathrm{ad}}, 1)$. Actually, this holds for a general Lie group (See 2.5 in [Jam08]).

Corollary 3.18 (Classical Equivariant Fundamental Groups)**.** *For a nilpotent orbit in a classical Lie algebra, $\pi_1(\mathscr{O})$ and $\mathscr{A}(\mathscr{O})$ are given in the following table.*

Algebra	$\pi_1(\mathscr{O}_{\mathbf{d}})$	$\mathscr{A}(\mathscr{O})$
$\mathfrak{sl}_n$	$\mathbb{Z}/c\mathbb{Z}$	$\{1\}$
$\mathfrak{so}_{2n+1}$	If $\mathbf{d}$ is rather odd, a central extension by $\mathbb{Z}/2\mathbb{Z}$ of $(\mathbb{Z}/2\mathbb{Z})^{a-1}$; otherwise, $(\mathbb{Z}/2\mathbb{Z})^{a-1}$	$(\mathbb{Z}/2\mathbb{Z})^{a-1}$
$\mathfrak{sp}_{2n}$	$(\mathbb{Z}/2\mathbb{Z})^{b}$	$(\mathbb{Z}/2\mathbb{Z})^{b}$ if all even parts have even multiplicity; otherwise $(\mathbb{Z}/2\mathbb{Z})^{b-1}$
$\mathfrak{so}_{2n}$	If $\mathbf{d}$ is rather odd, a central extension by $\mathbb{Z}/2\mathbb{Z}$ of $(\mathbb{Z}/2\mathbb{Z})^{\max\{0,a-1\}}$; otherwise $(\mathbb{Z}/2\mathbb{Z})^{\max\{0,a-1\}}$	$(\mathbb{Z}/2\mathbb{Z})^{\max\{0,a-1\}}$ if all odd parts have even multiplicity; otherwise $(\mathbb{Z}/2\mathbb{Z})^{\max\{0,a-2\}}$

Corollary 3.19. *Let $\mathfrak{g}$ be a semisimple Lie algebra of classical type and $\mathscr{O}$ a nilpotent orbit. Then $\mathscr{A}(\mathscr{O})$ is either trivial or a finite product of $\mathbb{Z}/2\mathbb{Z}$. In particular, it is always abelian.*

Proposition 3.20. *Let $\mathscr{O}_X$ be the adjoint orbit through any $X \in \mathfrak{g}$. Let $X = X_s + X_n$ be the Jordan decomposition of X. Then $\pi_1(\mathscr{O}_X)$ is isomorphic to the $G_{sc}^{X_s}$-equivariant fundamental group $\pi_1^{G_{sc}^{X_s}}(G_{sc}^{X_s} \cdot X_n)$ through X_n. In particular, every semisimple orbit in $\mathfrak{g}$ is simply connected.*

Proof. By the uniqueness of the Jordan decomposition and 3.9, we have
$$\begin{aligned}\pi_1(\mathscr{O}_X) &= G_{\mathrm{sc}}^X/(G_{\mathrm{sc}}^X)^\circ \\ &= (G_{\mathrm{sc}}^{X_s})^{X_n}/((G_{\mathrm{sc}}^{X_s})^{X_n})^\circ \\ &= \pi_1^{G_{\mathrm{sc}}^{X_s}}(G_{\mathrm{sc}}^{X_s} \cdot X_n)\end{aligned}$$
which proves the first statement. For the second assertion, see 2.3.3 in [CM93]. □

4. Explicit Standard Triples

Our next goal is to construct explicit standard triples in some classical Lie algebras. The main strategy is like this: Given a classical Lie algebra $\mathfrak{g}$, fix a choice of Cartan subalgebra $\mathfrak{h}$, together with a standard coordinate system on $\mathfrak{h}$. We will then write down all roots and root spaces of $\mathfrak{h}$ in $\mathfrak{g}$ and also fix a choice of positive roots. Now, given a partition $\mathbf{d}$, which we saw corresponds to a nilpotent orbit, we construct a standard triple $\{H, X, Y\}$ such that $H \in \mathfrak{h}$, X is a sum of vectors in certain positive root spaces and Y is a sum of certain vectors in negative root spaces. First, let's have a look at some

Root Space Decompositions.

- ⋆ Let $\mathfrak{g} = \mathfrak{sl}_n$. Denote by $\mathfrak{h}$ the set of diagonal matrices having trace zero. Recall the matrices E_{ij} having 1 at the (i, j)-th entry and zeros elsewhere. Let $e_i \in \mathfrak{h}^*$ with

$$e_i \begin{pmatrix} h_1 & & \\ & \ddots & \\ & & h_n \end{pmatrix} = h_i$$

We get that

$$(\operatorname{ad} H)E_{ij} = [H, E_{ij}] = (e_i(H) - e_j(H))E_{ij}$$

i.e. E_{ij} is a simultaneous eigenvector for all $\operatorname{ad}(H)$, with eigenvalue $e_i(H) - e_j(H)$. The $(e_i - e_j)$-root space is spanned by E_{ij} and we get a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathbb{C}E_{ij}$$

- ⋆ Let $\mathfrak{g} = \mathfrak{sp}_{2n}$. Remember that $\mathfrak{g}$ may be realised as the following set of matrices:

$$\{\begin{pmatrix} Z_1 & Z_2 \\ Z_3 & -Z_1^t \end{pmatrix} \mid Z_i \in \mathbb{C}^{n \times n}, Z_2, Z_3 \text{ symmetric}\}$$

Consider the Cartan subalgebra $\mathfrak{h}$ consisting of matrices of the form

$$H = \begin{pmatrix} h_1 & & & & & \\ & \ddots & & & & \\ & & h_n & & & \\ & & & -h_1 & & \\ & & & & \ddots & \\ & & & & & -h_n \end{pmatrix}$$

Let $e_j \in \mathfrak{h}^*$ be the linear functional taking a matrix H as above to its j-th entry. Then the root system of $\mathfrak{g}$ is given by

$$\Delta = \{\pm e_i \pm e_j \mid i \neq j\} \cup \{\pm 2e_k\}$$

As positive roots, we chose

$$\Phi = \{e_i \pm e_j, 2e_k \mid i \neq j\}$$

The root space decomposition is

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathbb{C}E_\alpha$$

With E_α defined as below. Let $\alpha \in \{\pm e_i \pm e_j, 2e_k\}$. Then E_α is defined as one of the following matrices:

$$\begin{aligned} E_{e_i - e_j} &= E_{i,j} - E_{j+n,i+n} & E_{2e_k} &= E_{k,k+n} \\ E_{e_i + e_j} &= E_{i,j+n} - E_{j,i+n} & E_{-2e_k} &= E_{k+n,k} \\ E_{-e_i - e_j} &= E_{i+n,j} + E_{j+n,i} & & \end{aligned}$$

We will now proceed with the construction of standard triples for $\mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$. Given a partition $\mathbf{d}$, we will break up its parts into chunks, each consisting of one or two parts. We will attach a set of positive roots to each chunk in such a way, that positive roots attached to distinct chunks are orthogonal. Our nilpotent element X corresponding to $\mathbf{d}$ will be a sum of positive root vector, one for each chunk of $\mathbf{d}$.

Recipe 1 (Type A_n). Let $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathbf{d} \in \mathscr{P}(n)$. The chunks of $\mathbf{d}$ are just its parts, each repeated as often as its multiplicity. For each chunk $\{d_i\}$, choose a block of consecutive indices $\{N_i+1, \ldots, N_i+d_i\}$ in such a way that disjoint block are attached to distinct chunks. To every chunk $\{d_i\}$, attach the set of simple roots

$$C^+ = C^+(d_i) = \{e_{N_i+1} - e_{N_i+2}, \ldots, e_{N_i+d_i-1} - e_{N_i+d_i}\}$$

Note that for $d_i = 1$, C^+ is empty. For every simple root α in $C := \bigcup_i C^+(d_i)$, let X_α be an α-root vector and write $X = \sum_{\alpha\in C} X_\alpha$. By Lemma 3.2.6 in [CM93], there is $Y = \sum_{\alpha\in C} X_{-\alpha}$ and $H \in \mathfrak{h}$ such that $\{H, X, Y\}$ is a standard triple. We have

$$H = \sum_i H_{C(d_i)}$$

where

$$H_{C(d_i)} = \sum_{l=1}^{d_i} (d_i - 2l + 1) E_{N_i+l, N_i+l}$$

Recipe 2 (Type C_n). Given $\mathbf{d} \in \mathscr{P}_{-1}(2n)$, break it up into chunks of the following types: pairs $\{2r+1, 2r+1\}$ of equal odd parts and single even parts $\{2q\}$. Now attach sets of positive (but not necessarily simple) roots to each chunk C as follows. If $C = \{2q\}$, choose a block $\{j+1, \ldots, j+q\}$ of consecutive indices and let

$$C^+ = C^+(2q) = \{e_{j+1} - e_{j+2}, \ldots, e_{j+q-1} - e_{j+q}, 2e_{j+q}\}$$

If $C = \{2r+1, 2r+1\}$, choose a block $\{l+1, l+2r+1\}$ of consecutive indices and let

$$C^+ = C^+(2r+1, 2r+1) = \{e_{l+1} - e_{l+1}, \ldots, e_{l+2r} - e_{l+2r+1}\}$$

We further require that the blocks attached to distinct chunks be disjoint. However, this does not impose any restriction. For example, if $\mathfrak{g} = \mathfrak{sp}_{20}$ and $\mathbf{d} = [6, 5^2, 2^2]$, then its chunks are $\{6\}, \{5, 5\}, \{2\}$ and $\{2\}$ and we may take

$$\begin{aligned} C^+(6) &= \{e_1 - e_2, e_2 - e_3, 2e_3\} \\ C^+(5,5) &= \{e_4 - e_5, e_5 - e_6, e_6 - e_7, e_7 - e_8\} \\ C^+(2) &= \{2e_9\} \\ C^+(2) &= \{2e_{10}\} \end{aligned}$$

Let $C = \bigcup_i C^+(d_i)$ and once again define $X = \sum_{\alpha\in C} X_\alpha$. Then there is a sum $Y = \sum_{\alpha\in C} X_{-\alpha}$ and $H \in \mathfrak{h}$ such that $\{H, X, Y\}$ is a standard triple. We have

$$H = \sum_C H_C$$

where

$$H_C = \sum_{l=1}^{q} (2q - 2l + 1)(E_{j+l,j+l} - E_{n+j+l,n+j+l})$$

if $C^+ = \{e_{j+1} - e_{j+2}, \ldots, e_{j+q-1} - e_{j+q}, 2e_{j+q}\}$ and

$$H_C = \sum_{m=0}^{2r} (2r - 2m)(E_{l+1+m,l+1+m} - E_{n+l+1+m,n+l+1+m})$$

if $C^+ = \{e_{l+1} - e_{l+2}, \ldots, e_{l+2r} - e_{l+2r+1}\}$

Proposition 4.1. *If we view $\mathfrak{sp}_{2n}$ as a subalgebra of $\mathfrak{sl}_{2n}$, then the partition attached to the standard triple $\{H, X, Y\}$ is $\boldsymbol{d}$.*

Proof. See [CM93]. □

References

[Hum72] J.E. Humphreys. *Introduction to Lie Algebras and Representation Theory*. Graduate texts in mathematics. Springer, 1972. ISBN: 9780387900537. URL: https://books.google.de/books?id=TiUlAQAAIAAJ.

[Hes76] Wim Hesselink. "Singularities in the Nilpotent Scheme of a Classical Group". In: *Transactions of the American Mathematical Society* 222 (1976), pp. 1–32. ISSN: 00029947. URL: http://www.jstor.org/stable/1997656 (visited on 12/11/2023).

[Spa82] N. Spaltenstein. *Classes Unipotentes et Sous-groupes de Borel*. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 1982. ISBN: 9783540115854. URL: https://books.google.de/books?id=l4vdRwAACAAJ.

[FH91] W. Fulton and J. Harris. *Representation Theory: A First Course*. Graduate texts in mathematics. Springer, 1991. ISBN: 9783540974956. URL: https://books.google.de/books?id=jW6hngEACAAJ.

[Bre93] G.E. Bredon. *Topology and Geometry*. Graduate Texts in Mathematics. Springer, 1993. ISBN: 9780387979267. URL: https://books.google.de/books?id=G74V6UzL_PUC.

[CM93] D.H. Collingwood and W.M. McGovern. *Nilpotent Orbits In Semisimple Lie Algebra: An Introduction*. Mathematics series. Taylor & Francis, 1993. ISBN: 9780534188344. URL: https://books.google.de/books?id=9qdwgNmjLEMC.

[Jam08] Juan Pablo-Ortega James Montaldi. *Notes on lifting group actions*. MIMS EPrint, 2008.