DYNKIN CLASSIFICATIONS

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In this seminar, we aim to categorize irreducible root systems, up to isomorphism, through Dynkin diagrams. Our goal is to prove that every root system Φ, of rank l, is characterized precisely, up to isomorphism, by a Dynkin diagram of the form A_l, B_l, C_l, D_l, E_l, F_l, G_l, these Dynkin diagrams will be defined later in this handout.

To achieve this, we follow [Hum72, Section 11], and sketch a proof of the classification theorem [Hum72, Theorem 11.4] at the end.

Notation: k is an algebraically closed field of characteristic 0.

Warning: The numbering and ordering in this handout may not be the same as that of my presentation.

1. Root systems

Definition 1.1. Let V be a rational finite-dimensional vector space, of rank l, with inner-product $(\_,\_)$. A subset $\Phi \subset V$ is a *(reduced) root system*, of rank l, if Φ fulfills the following:

(i) Φ is finite, does not contain 0, and spans V.
(ii) For all $\alpha \in \Phi$, the integer multiples of α in Φ are precisely $\alpha \in \Phi$ and $-\alpha \in \Phi$.

For any nonzero element $\alpha \in V$, we define the *reflection* $s_\alpha : V \to V$:

$$s_\alpha(x) = x - 2\frac{(x,\alpha)}{(\alpha,\alpha)}\alpha, \tag{1}$$

along the axis of α.

(iii) For every $\alpha \in \Phi$, we have that s_α preserves Φ, i.e., $s_\alpha(\Phi) \subset \Phi$.
(iv) For all $\alpha, \beta \in \Phi$, $s_\alpha(\beta) - \beta$ is an integer multiple of α.

What is this definition useful for, and what does it model? As a reminder from Till's talk, this is the main example of a root system.

Theorem 1.1. *Let $\mathfrak{t}$ be a Cartan subalgebra of a semisimple Lie algebra $\mathfrak{g}$, and let $\Phi(\mathfrak{g},\mathfrak{t})$ be the roots of the corresponding root space decomposition of $\mathfrak{g}$.*

$\Phi(\mathfrak{g},\mathfrak{t})$ is contained in a rational vector space V, whose extension of scalars to k is isomorphic to $\mathfrak{t}^\vee = Hom_k(\mathfrak{t}, k)$. We equip V with an inner-product $(\_,\_)$ induced by the Killing-form κ of $\mathfrak{g}$.

We claim that $\Phi(\mathfrak{g},\mathfrak{t})$ is a root system of V, with respect to $(\_,\_)$.

Any constructions and statements on abstract root systems can then be applied to the special case of a semisimple Lie algebra $\mathfrak{g}$. This leads to a rich theory of Lie algebras, Lie groups, and algebraic groups, that is further explored in [Hum72], [Kna88], and more.

Unless otherwise stated, $\Phi \subset V$ is a root system, with respect to $(\_,\_)$.

Definition 1.2. (a) A choice of half the roots $\Phi^+ \subset \Phi$ is a set of *positive roots* of Φ, if we have the following:

(i) For all $\alpha \in \Phi$, we have $\alpha \in \Phi^+$ or $-\alpha \in \Phi^+$, but never both.
(ii) For all $\alpha, \beta \in \Phi^+$, such that $\alpha + \beta \in \Phi$, we have $\alpha + \beta \in \Phi^+$.

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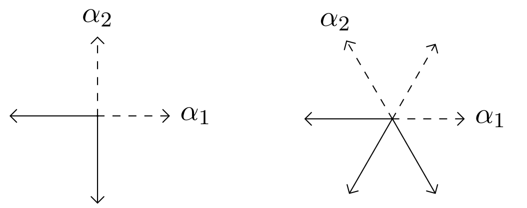
(b) Given positive roots $\Phi^+ \subset \Phi$, there exists a unique subset $\triangle \subset \Phi^+$ of *simple roots*, such that $\triangle$ fulfills the following:
  (i) Every root in $\Phi^+ \setminus \triangle$ is a nonnegative integer linear combination of roots in $\triangle$.
  (ii) $\triangle$ is a minimal set that fulfills (i), with respect to inclusion. Equivalently, $\triangle$ forms a basis of V.

Example 1.1. If we imagine V to be $\mathbb{Q}^2$ with the standard euclidean inner-product $(\_,\_)$, the following depict root systems:



(2)

Left: Root system $A_1 \times A_1$, of the Lie algebra $\mathfrak{sl}(2,k) \oplus \mathfrak{sl}(2,k)$.
Right: Root system A_2, of the Lie algebra $\mathfrak{sl}(3,k)$.
The dashed arrows depict a valid choice of positive roots, and the labeled dashed arrows are the corresponding simple roots.

Remark 1.1. (a) $W = \langle s_\alpha | \alpha \in \Phi \rangle \subset \mathrm{Aut}_{\mathbb{Q}}(V)$ is the *Weyl group* of Φ, which is finite and acts transitively on the Weyl chambers of Φ.
(b) For $\alpha, \beta \in \Phi$, denote $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)} \in V$ as the *dual root*, and $\langle \beta, \alpha \rangle = (\beta, \alpha^\vee) \in \mathbb{Z}$.
(c) For $\alpha, \beta \in \Phi$, denote the induced norm of $(\_,\_)$ by $||\_||$, then the angle formula $||\alpha|| ||\beta|| \cos(\theta) = (\alpha, \beta)$ holds, where θ describes the angle between α and β.
With some manipulation, one proves that $\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle = 4(\cos(\theta)^2)$. As $\cos(\theta) \in [0,1]$, and as $\langle \beta, \alpha \rangle, \langle \alpha, \beta \rangle \in \mathbb{Z}$, it is easy to verify that $\langle \alpha, \beta \rangle \in \{-3,-2,-1,0,1,2,3\}$.
Assuming $\alpha \neq \pm\beta$, and $||\beta|| \geq ||\alpha||$, we obtain the only possibilities, as seen in [Hum72, Section 9.4]:

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	θ	$\|\|\beta\|\|^2/\|\|\alpha\|\|^2$
0	0	$\pi/2$	no relation
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

(3)

(d) For positive roots $\Phi^+ \subset \Phi$, with simple roots $\triangle$, and for any $\alpha, \beta \in \triangle$, $\alpha \neq \beta$, we find that $\langle \alpha, \beta \rangle, (\alpha, \beta) \leq 0$, i.e., the angle between α and β is obtuse, or a right angle. This is proven in [Hum72, Lemma 10.1].
Thus, only the odd rows of (3) occur for $\langle \alpha, \beta \rangle$ and $\langle \beta, \alpha \rangle$.

2. Cartan matrices

From Philipp's talk, we learned about Cartan matrices and Dynkin diagrams.

As usual, $\Phi \subset V$ is a root system, with respect to $(\_,\_)$, with fixed positive roots Φ^+ and simple roots $\triangle = \{\alpha_1, \ldots, \alpha_l\}$. $\Phi' \subset V'$, is a second root system with Φ'^+, and $\triangle' = \{\alpha'_1, \ldots, \alpha'_{l'}\}$.

Definition 2.1. (a) The matrix $C = (\langle \alpha_i, \alpha_j \rangle)_{i,j}$ is called a *Cartan matrix* of Φ.

The entries of C only takes values in $-3,-2,-1,0,2$, where precisely only the diagonals of C take the value 2, this is due to (3).

Taking a different ordering of $\triangle$ results in a different matrix C, i.e., C is not uniquely determined by Φ alone.

In the same situation as (a), we can try to encode $\langle\alpha_i,\alpha_j\rangle$ through edges and nodes.

(b) Fix $\{\alpha_i,\alpha_j\}$, $i\neq j$, and observe the nodes $\circ\ \ \circ$, where α_i corresponds to one node, and α_j to the other. We assign edges to $\circ\ \ \circ$ as follows:
   (i) If $\langle\alpha_i,\alpha_j\rangle=0$, i.e., they are orthogonal, we assign no edges.
   (ii) If row three occurs in (3), assign one edge.
   (iii) If row five occurs in (3), assign two edges.
   (iv) If row seven occurs in (3), assign three edges.

   By repeating the same process for all $\{\alpha_i,\alpha_j\}_{i,j=1,\ldots,l,i\neq j}$, we obtain a graph with l nodes and some edges, this is the *coexeter graph* of Φ.

   Unlike Cartan matrices, this does not depend on the choice of positive roots and labeling of simple roots.

Can a Cartan matrix of Φ be recovered from the coxeter graph, when the nodes are unlabeled? Almost.

(c) In the situation of the nodes in (b), where we have two nodes with a double or triple edge, corresponding to α_i and α_j, we still do not know which simple root has the longer norm, as seen in rows three and five of (3). By adding to such double and triple edges an arrow pointing to the node with the root with the smaller norm, we have enough information to find a Cartan matrix of Φ.

   The coexeter graph of Φ, with these arrows, is called the *Dynkin diagram* of Φ, and does not depend on the choice of positive roots and labeling of simple roots.

Example 2.1. Observe the root system $\Phi=A_2$, as seen in the right diagram of (2) in Example 1.1, we calculate a Cartan matrix C, and the Dynkin diagram:

$$C=\begin{pmatrix}2&-1\\-1&2\end{pmatrix},\quad \circ\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\circ\ . \tag{4}$$

Why do we study Cartan matrices and Dynkin diagrams? What does it encode about Φ, and how does it help characterize Φ up to isomorphism? These questions, which are important for the classification theorem, are addressed in the following lemma.

Lemma 2.1. *Let there be a bijection $\alpha_i\mapsto\alpha_i'$ between simple roots in $\triangle$ and $\triangle'$, preserving the Cartan matrices of Φ and Φ', i.e., for all $i,j=1,\ldots,l=l'$, we have $\langle\alpha_i,\alpha_j\rangle=\langle\alpha_i',\alpha_j'\rangle'$. This bijection extends uniquely to an isomorphism $\varphi:V\to V'$ of vector spaces such that:*
(i) For all $\alpha,\beta\in\Phi$, we have $\langle\varphi(\alpha),\varphi(\beta)\rangle'=\langle\alpha,\beta\rangle$.
(ii) For all $\alpha\in\Phi$, we have $\varphi\circ s_\alpha=s'_{\varphi(\alpha)}\circ\varphi$.

Proof. See [Hum72, Proposition 11.1]. □

The properties (i) and (ii) in Lemma 2.1 formalize the concept of two root systems being isomorphic to each other. It is clear that two isomorphic root systems share the same Cartan matrices and Dynkin diagrams.

In these isomorphism classes, the labeling of the simple roots α_i becomes irrelevant, and a Cartan matrix C determines an isomorphism class of root systems. Furthermore, isomorphism classes of root systems and Dynkin diagrams are in bijection with each other.

Definition 2.2. A root system Φ of V is called *irreducible* if there exists no orthogonal partition of Φ, i.e., no $A\sqcup B=\Phi$, such that for all $\alpha\in A$ and $\beta\in B$, we have $\langle\alpha,\beta\rangle=(\alpha,\beta)=0$. Equivalently, Φ is irreducible if its coexeter graph is connected.

Remark 2.1. Every root system Φ of V decomposes into disjoint subsets $\Phi = \Phi_1 \sqcup \ldots \sqcup \Phi_t$, where each Φ_i is an orthogonal component of Φ, and where for all $i = 1, \ldots, t$, Φ_i is an irreducible root system of a subspace V_i of V, such that $V = V_1 \oplus \ldots \oplus V_t$.

Any choice of positive roots Φ_i^+, for all $i = 1, \ldots, t$, corresponds to a choice of positive roots $\Phi^+ = \Phi_1^+ \sqcup \ldots \sqcup \Phi_t^+$ of Φ. Furthermore, with respect to these positive roots, we have for the simple roots $\triangle_i$ of Φ_i, $i = 1, \ldots, t$, that the simple roots of Φ are $\triangle = \triangle_1 \sqcup \ldots \sqcup \triangle_t$.

Let us apply these statements to semisimple Lie algebras $\mathfrak{g}$, through the use of Theorem 1.1. As a reminder, $\mathfrak{g}$ is simple if $\mathfrak{g}$ admits no nontrivial proper ideals. By fixing a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, we induce a root system $\Phi(\mathfrak{g}, \mathfrak{t})$ using Theorem 1.1. By observing the root space decomposition of $\mathfrak{g}$, with respect to $\Phi(\mathfrak{g}, \mathfrak{t})$, it can be shown that $\mathfrak{g}$ being simple is equivalent to $\Phi(\mathfrak{g}, \mathfrak{t})$ being irreducible.

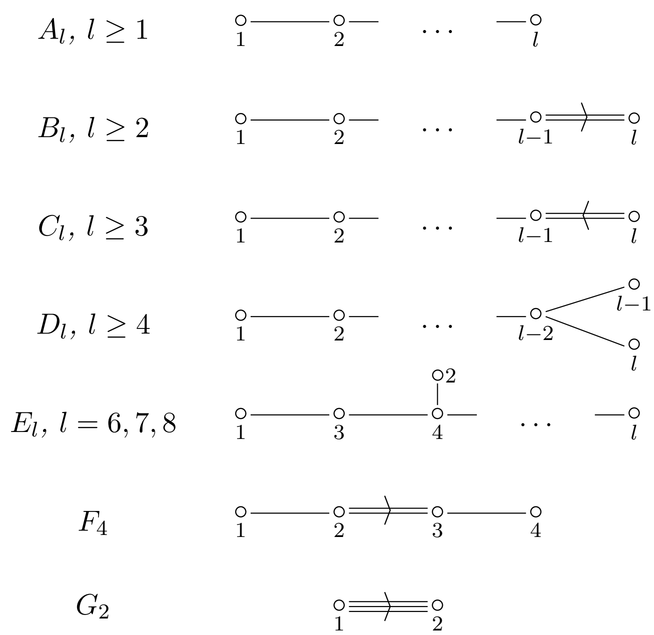
Every semisimple Lie algebra $\mathfrak{g}$ is the direct sum of nontrivial proper ideals $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_t$, furthermore, $\mathfrak{t}$ is the direct sum of Cartan subalgebra $\mathfrak{t}_i$ of $\mathfrak{g}_i$, i.e., $\mathfrak{t} = \mathfrak{t}_1 \oplus \ldots \oplus \mathfrak{t}_t$.

We have that $\Phi(\mathfrak{g}, \mathfrak{t})$, its positive roots, and its simple roots, respectively, are disjoint unions of $\Phi(\mathfrak{g}_i, \mathfrak{t}_i)$, their positive roots, and their simple roots, for all $i = 1, \ldots, t$, respectively.

The classification theorem of irreducible root systems, which is Theorem 3.1, will thus also characterize simple Lie algebras $\mathfrak{g}$ up to isomorphism, and thus also indirectly semisimple Lie algebras, as they are finite direct sums of simple Lie algebras.

3. The classification theorem

Theorem 3.1. *Let Φ be an irreducible root system of V, of rank l, with respect to the inner-product $(\_, \_)$. The Dynkin diagram of Φ, which characterizes Φ up to isomorphism, must be one of the following:*



(5)

The restrictions on l are chosen such that no Dynkin diagram appears more than once.

Remark 3.1. A_l corresponds to root systems of $\mathfrak{sl}(l+1, k)$, B_l corresponds to root systems of $\mathfrak{so}(2l+1, k)$, C_l corresponds to root systems of $\mathfrak{sp}(2l, k)$, D_l corresponds to root systems of $\mathfrak{so}(2l, k)$. The diagrams E_l, F_4, and G_2 are special cases, but also have corresponding exotic simple Lie algebras. For example, F_4 is the Dynkin diagram of the Lie algebra $\mathfrak{f}_4$, which is the complexification of a Lie algebra of a Lie group, also named F_4, which is the isometry group of a projective space of octonions!

Due to the isomorphisms of diagrams $A_1 \simeq B_1 \simeq C_1$, we follow that $\mathfrak{sl}(2,k) \simeq \mathfrak{so}(3,k) \simeq \mathfrak{sp}(2,k)$. Similar isomorphisms of Lie algebras are induced from the isomorphisms of diagrams $B_2 \simeq C_2$, $D_2 \simeq A_1 \times A_1$, $D_3 \simeq A_3$.

Proof. This will be a rather long, multistep proof, so we will start with some simplifications. Firstly, let us only classify the underlying coexeter graphs of irreducible root systems. Achieving this would almost provide the proof, where we would only then have to make considerations about B_l and C_l, which are the only differing Dynkin diagrams with the same underlying coexeter graphs.

By only considering coexeter graphs, we firstly disregard the lengths of roots, and only consider angles between roots, so we firstly work with normed vectors. The following definition should model vectors that are normed vectors of simple roots in $\triangle$.

Definition 3.1. Let $n \leq l$, If $A = \{v_1, \ldots, v_n\} \subset V$, such that:

(i) A is linearly independent.
(ii) For all $i = 1, \ldots, n$, we have $||v_i|| = 1$.
(iii) For all $i, j = 1, \ldots, n$, $i \neq j$, we have $(v_i, v_j) \leq 0$.
(iv) For all $i, j = 1, \ldots, n$, $i \neq j$, we have $4(v_i, v_j)^2 = 0, 1, 2, 3$.

We then call A *admissible.*

Note that simple roots fulfill (iii) due to (d) of Remark 1.1, and they fulfill (iv) due to (c) of Remark 1.1, and its table in (3).

We denote the coexeter graph induced by A as Γ_A. Note that subsets A' of admissible sets A are admissible, and thus generate coexeter subgraphs $\Gamma_{A'}$ of Γ_A.

Let $A = \{v_1, \ldots, v_n\} \subset V$ be an admissible set.

Claim 1: The number of two nodes in Γ_A with at least an edge between them is less than n.

Proof of Claim 1: Let $v = \sum_{i=1}^{n} v_i$, v is nonzero as A is linearly independent. If there is an edge between the nodes of v_i and v_j, $i \neq j$, we equivalently have $4(v_i, v_j)^2 = 1, 2, 3$, due to (c) of Remark 1.1, which is equivalent to $2(v_i, v_j) \leq -1$. Hence, due to:

$$0 < (v, v) = n + 2 \sum_{i<j\leq n} (v_i, v_j), \tag{6}$$

the claim follows. #

Claim 2: Γ_A contains no cycles (or loops).

Proof of Claim 2: If there exists an admissible subset $A' \subset A$, whose subgraph $\Gamma_{A'}$ of Γ_A is a cycle, A' would violate Claim 1, as A' has the same number of nodes as edges. Thus, the claim follows. #

Claim 3: No more than three edges can originate from a node of Γ_A.

Proof of Claim 3: Fix a vector $v \in A$, and let $\{a_1, \ldots, a_s\} \subset A$, $s \leq n$, be precisely the vectors whose nodes have an edge with v in Γ_A. Since Γ_A cannot contain cycles, for all $i, j = 1, \ldots, s$, $i \neq j$, a_i and a_j do not share an edge, and thus $(a_i, a_j) = 0$.

We find $a_0 \in \operatorname{span}_k(v, a_1, \ldots a_s)$, $a_0 \perp \operatorname{span}_k(a_1, \ldots a_s)$, $||a_0|| = 1$. Then due to $(v, a_0) \neq 0$, we have:

$$\left(v = \sum_{i=0}^{s} (v, a_i) a_i \right) \Rightarrow \left(\sum_{i=0}^{s} (v, a_i)^2 = (v, v) = 1 \right) \Rightarrow \left(\sum_{i=1}^{s} 4(v, a_i)^2 < 4 \right). \tag{7}$$

Just as in the proof of Claim 1, the edges between v and a_i imply that $4(v, a_i)^2 = 1, 2, 3$, and hence $s \leq 3$. Thus, there can only be at most three edges going to the node in Γ_A corresponding to v. #

As a consequence, the only connected coexeter graph Γ_A with a triple edge is the underlying graph of G_2.

Claim 4: Let $s \leq n$, $B = \{v_1, \dots, v_s\} \subset A$ be a subset, whose subgraph Γ_B of Γ_A is of the form A_s, and let $v = \sum_{i=1}^{s} v_i$. We claim that $A' = (A \setminus B) \cup \{v\}$ is admissible.

Proof of Claim 4: By construction, A' is clearly linearly independent, so we have (i) of Definition 3.1.

To check that $||v|| = 1$, to verify (ii) of Definition 3.1, we note that the only non-orthogonal pairs of vectors in B are $\{v_i, v_{i+1}\}$, for $i = 1, \dots, s-1$, and we have $2(v_i, v_{i+1}) = -1$, then:

$$(v, v) = s + 2 \sum_{i<j\leq s} (v_i, v_j) = s - (s-1) = 1. \tag{8}$$

For all $a \in A'$, $a \neq v$, if a is orthogonal to v, then (iii) and (iv) of Definition 3.1 follow.

Otherwise, we have $(a, v) < 0$, as A is admissible. The fact that Γ_A contains no cycles, from Claim 3, ensures that a can only have edges going into at most one element of $B = \{v_1, \dots, v_s\}$. Hence, there exists one unique $i = 1, \dots, s$, such that $(a, v) = (a, v_i) < 0$. From this, it follows that $4(a, v)^2 = 1, 2, 3$, due to the admissibility of A, and (iii) and (iv) of Definition 3.1 follow. #

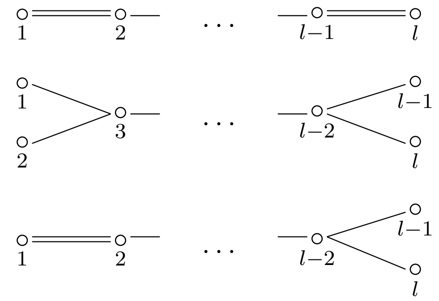
Note that in the situation of Claim 4, we obtain $\Gamma_{A'}$ from Γ_A by collapsing the subgraph $\Gamma_B \simeq A_s$ of Γ_A into a single node $\circ$, corresponding to v. Any edge between $\Gamma_A \setminus \Gamma_B$ and Γ_B is then replaced by an edge between $\Gamma_A \setminus \Gamma_B$ and the node of v.

Claim 5: Γ_A contains no subgraph of the form:

5(a)

5(b)

5(c)



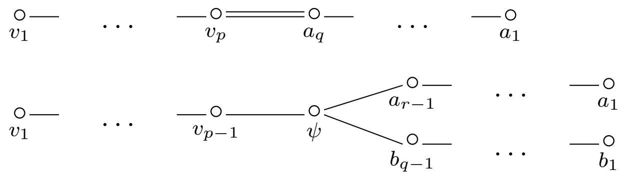
(9)

Proof of Claim 5: Assuming to the contrary, from each of these coexeter subgraphs, we form a new coexeter graph by collapsing their middle sections $\circ - \dots - \circ$ to a single node $\circ$. These new graphs correspond to admissible sets, due to Claim 4, with a node in the middle with 4 edges, in contradiction to Claim 3. Hence, the claim follows. #

Claim 6: For any admissible set A with a connected graph Γ_A, Γ_A is of the form 6(a) A_l, or 6(b) G_2 (its underlying graph), or:

6(c)

6(d)



(10)

Proof of Claim 6: Assuming Γ_A has a triple edge, then only 6(b) is possible due to Claim 3.

If Γ_A contains no triple edges, and at contains least one double edge, then the forbidden graphs of 5(a) and 5(c) leave us with the only possibility being 6(c).

If Γ_A contains only single edges, then the forbidden graph 5(b) leaves us with only A_l and 6(d). #

Claim 7: If A has a connected graph Γ_A, such that Γ_A is of the form 6(c) from Claim 6, we have that Γ_A is of the form 7(a) F_4 (its underlying graph), or 7(b) B_l, C_l (its underlying graph).

Proof of Claim 7: Let $v = \sum_{i=1}^{p} iv_i$ and $a = \sum_{i=1}^{q} ia_i$, then since the only non-orthogonal pairs in $\{v_1, \dots, v_p\}$ are $\{v_i, v_{i+1}\}$, for $i = 1, \dots, p-1$, with $2(v_i, v_{i+1}) = -1$, we get:

$$(v, v) = \sum_{i=1}^{p} i^2 - \sum_{i=1}^{p-1} i(i+1) = \frac{p(p+1)}{2}, \tag{11}$$

and analogously $(a, a) = q(q+1)/2$.

Since $4(v_p, a_q)^2 = 2$, we also have $(v, a)^2 = p^2q^2(v_p, a_q)^2 = p^2q^2/2$.

As a and v are linearly independent, we have due to the Cauchy-Schwarz inequality that $(v, a)^2 < (v, v)(a, a)$. By entering the expressions we found for each term, we get an inequality for p and q that simplifies to $(p-1)(q-1) < 2$.

Thus, the only possibilities are $p = q = 2$, where we obtain $\Gamma_A \simeq F_4$ as graphs, or otherwise $p = 1$ or $q = 1$, where we obtain $\Gamma_A \simeq B_l \simeq C_l$ as graphs. #

Claim 8: If A has a connected graph Γ_A, such that Γ_A is of the form 6(d) from Claim 6, we have that Γ_A is of the form 8(a) D_l, or 8(b) E_l.

Proof of Claim 8: Let $v = \sum_{i=1}^{p-1} iv_i$, $a = \sum_{i=1}^{r-1} ia_i$, and $b = \sum_{i=1}^{q-1} ia_i$, then we get $(v, v) = p(p-1)/2$, $(a, a) = r(r-1)/2$, and $(b, b) = q(q-1)/2$, just like in Claim 7. Since v, a, and b are pairwise orthogonal and linearly independent, we can apply the same argument as in (7) to the normed vectors of v, a, and b to obtain:

$$\frac{(v, \psi)^2}{(v, v)} + \frac{(a, \psi)^2}{(a, a)} + \frac{(b, \psi)^2}{(b, b)} < 1. \tag{12}$$

Using the definitions of v, a, and b, and using how the graph 6(d) is constructed, it is easy to calculate that the first term of (12) is $(1 - 1/p)/2$, the second is $(1 - 1/r)/2$, the third is $(1 - 1/q)/2$. Plugging this into (12), we obtain:

$$\frac{1}{p} + \frac{1}{r} + \frac{1}{q} > 1. \tag{13}$$

We may assume without loss of generality that $p \geq q \geq r$. If $p = q = r = 1$, then we obtain $\Gamma_A \simeq D_3 \simeq A_3$. Otherwise, $p, q \geq 2$ implies $r = 2$, and we either have any $p \geq 2$, with $q = r = 2$, obtaining the graph D_l, or otherwise $p = 3, 4, 5$, with $q = 3$ and $r = 2$, obtaining the graphs E_6, E_7, and E_8. #

Conclusion: Until now, we have proven that the only possible connected Dynkin systems are those with the underlying coexeter graphs A_l, B_l, C_l, D_l, E_l, F_l, G_l. By explicitly finding Dynkin systems of root systems with these diagrams, as mentioned in Remark 3.1, we constructively verify the existence of such root systems. This is especially important in the cases of B_l and C_l, where they have the same underlying coexeter graph, but are the Dynkin systems of two slightly different root systems. □

4. References

Books.

[Hum72] James Humphreys. *Introduction to Lie Algebras and Representation Theory.* Springer, 1972. URL: `https://www.math.uci.edu/~brusso/humphreys.pdf`.

[Kna88] Anthony Knapp. *Lie Groups Beyond an Introduction: Second Edition.* Birkhäuser, 1988. URL: `https://www.math.stonybrook.edu/~aknapp/books/green/beyond2-frontmatter.pdf`.