Cartan matrices and Dynkin diagrams

Philipp Müller

December 2023

One goal of the seminar is to prove the classification theorem for semisimple Lie algebras. In the previous talks we have seen that we can reduce this question to the question of classifying the simple Lie algebras and that we can attach to each simple Lie algebra, after choosing a Cartan subalgebra, an unique (irreducible) root system by using the root system decomposition. Therefore we get an injective map (up to isomorphism):

$$\text{(simple Lie algebras, Cartan subalgebra)} \;\rightarrow\; \text{irreducible root systems.}$$

The aim of this talk is to reduce the question of classification further to the question of classifying so called Dynkin diagrams, by showing the $1:1$ correspondence (up to isomorphism):

$$\text{irreducible root systems} \;\leftrightarrow\; \text{Dynkin diagrams.}$$

1. Notation

In this talk V will denote a real vector space and $V^* := \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R})$ the dual space.

2. Recap

Recall that we have defined a subset R in V to be a **root system**, if

1. R is finite, $\operatorname{span}(R) = V$ and $0 \notin R$,
2. For each $\alpha \in R$, there exists a symmetry $s_\alpha \in \mathrm{GL}(V)$ leaving R invariant,
3. For each $\alpha, \beta \in R$, we have $s_\alpha(\beta) - \beta \in \alpha \cdot \mathbb{Z}$.

We call a root system R **reduced**, if for each $\alpha \in R$, the elements $\alpha, -\alpha$ are the only roots proportional to α. We say a subset S of R is a **base** for R if

1. S is a basis for V,
2. Each root $\beta \in R$ can be written as $\mathbb{Z}$-linear combination of elements in S, s.t. all coefficients have the same sign.

Write $R = R^+ \cup R^-$, with $R^+ = \{\text{roots with non negative coefficients w.r.t. S}\}$ and $R^- = -R^+$. We know, that there is always a base, see Theorem A.4. Finally, we call the normal subgroup

$$W(R) := \langle s_\alpha \mid \alpha \in R \rangle \triangleleft \operatorname{Aut}(R)$$

the **Weyl group** of R. We endow V with a positive definite symmetric bilinear form $(-,-)$, which is invariant under $W(R)$.

Remark 2.1. Let R be a root system. We define it's dual system, as $R^* = \{\alpha^* \mid \alpha \in R\}$, where α^* is the unique element in V^*, s.t. α^* defines the symmetry s_α. We can then identify $W(R)$ and $W(R^*)$ as groups via $w \mapsto {}^t w^{-1}$. This group isomorphism preserves the group action of $W(R)$ in the following sense: Let $w' \in W(R^*)$ and $w \in W(R)$ its preimage, then $w' \cdot t^*(-) = t^*({}^t w^{-1}(-))$ for all $t^* \in V^*$.

3. Properties of the Weyl group

From now on let R be a reduced root system, and S a base for R. We have the following

Theorem 3.1. *1. For each $t^* \in V^*$, there exists a $w \in W(R)$, s.t. $w \cdot t^*(\alpha) \geq 0$, for all $\alpha \in S$.*

2. If S' is another base for R, there exists a $w \in W(R)$, s.t. $w(S') = S$, i.e. $W(R)$ acts transitively on the set of bases of R.

3. For each $\beta \in R$, there exists a $w \in W(R)$, s.t. $w(\beta) \in S$, i.e. S is a fundamental domain of the action of $W(R)$ on R.

4. $W(R) = \langle s_\alpha \mid \alpha \in S \rangle =: W_S$.

Proof. We will show 1.-3. for W_S instead of $W(R)$. Since $W_S \subset W(R)$, that will show the statements.

1. Let $t^* \in V^*$ and $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$. Since W_S is finite, we can find an element $w \in W_S$, s.t. $w = \operatorname{argmax}\{w \cdot t^*(\rho) \mid w \in W_S\}$. In particular, we have

$$w \cdot t^*(\rho) \geq (s_\alpha w) \cdot t^*(\rho)$$

for all $\alpha \in S$. With Remark 2.1 and the fact that s_α is of order 2 and symmetric we conclude:

$$w \cdot t^*(\rho) \geq (s_\alpha w) \cdot t^*(\rho) = w \cdot t^*(s_\alpha(\rho)) = w \cdot t^*(\rho) - w \cdot t^*(\alpha).$$

In the last equation we use Lemma A.5. Hence we can conclude $w \cdot t^*(\alpha) \geq 0$ for all $\alpha \in S$.

2. Let $t'^* \in V^*$ be an element, s.t. $t'^*(\alpha') > 0$ for all $\alpha' \in S'$. (Such an element always exists, for instance choose the sum over the dual base of S'.) By 1. we find a $w \in W_S$, s.t. $w \cdot t'^* =: t^*$ satisfies $t^*(\alpha) \geq 0$ for all $\alpha \in S$. Using Remark 2.1 again we see

$$t^*(\alpha) = t'^*(w^{-1}(\alpha))$$

and since w^{-1} leaves R invariant we can write $w^{-1}(\alpha) = \sum_{\alpha' \in S'} m_{\alpha'}\alpha'$ with $m_{\alpha'} \geq 0$ and not all equal to 0. By the choice of t'^* it follows, that $t^*(\alpha) > 0$. Using Theorem A.4 we have

$$S = S_{t^*} \text{ and } S' = S_{t'^*}.$$

The statement follows since $S_{w(t^*)} = w(S_{t^*})$.

3. Let $\beta \in R$ and define $L_\beta := \{v \in V \mid (v, \beta) = 0\}$. It is an easy consequence, that given $\alpha, \beta \in R$ we have $L_\beta = L_\alpha \Leftrightarrow \beta \in \alpha\mathbb{R}$. Another basic fact from linear algebra is, that a finite dimensional vector space over a infinite field is never the union of finitely many proper subspaces, therefore there is an element $t_0 \in L_\beta$, s.t. $t_0 \notin L_\alpha$ for $\alpha \neq \pm\beta$. Using the continuity of $(-, \beta)$ we find an element $t \in V$ with

$$(t, \beta) = \varepsilon > 0 \text{ and } |(t, \alpha)| > \varepsilon \text{ for all } \alpha \neq \pm\beta$$

since $|(t, \alpha)|$ β is indecomposable and therefore an element of S_t. The statement follows with 2.

4. By definition we only need to show, that $s_\beta \in W_S$ for all $\beta \in R$. Using 3. we find a $w \in W_S$, s.t. $\alpha = w(\beta)$ belongs to S. The statement now follows by the formula

$$s_\beta = w^{-1} s_\alpha w.$$

□

4. Cartan matrix

In this section we introduce the notion of a Cartan matrix associated to a reduced root system. We then show, that a Cartan matrix determines a root system (up to isomorphism).

Definition 4.1. *Let R be a reduced root system and S a base for R. We call the matrix*

$$C(S) := (n(\alpha, \beta))_{\alpha,\beta \in S}$$

*the **Cartan matrix** of R (with respect of the base S). Here $n(\alpha, \beta) = 2\frac{(\beta,\alpha)}{(\beta,\beta)}$ is the Cartan integer of α and β.*

Lemma 4.2. *The Cartan matrix of a root system is (up to order) independent of the chosen base.*

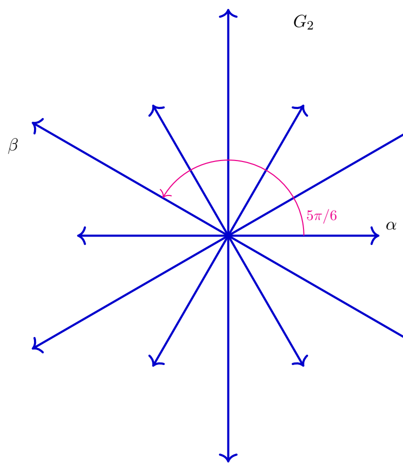
Proof. Let S, S' be two bases of R. By Theorem 3.1 we find $w \in W(R)$, s.t. $S = w(S')$. Therefore the following equations have to be true:

$$C(S) = (n(\alpha, \beta))_{\alpha,\beta \in S} = (n(w(\alpha'), w(\beta')))_{\alpha',\beta' \in S'} = (n(\alpha', \beta'))_{\alpha',\beta' \in S'} = C(S').$$

The second to last equation is a direct consequence of the invariance of $(-, -)$ with respect to $W(R)$. □

Remark 4.3. As a consequence of Lemma 4.2 we write $C(R)$ instead of $C(S)$ and call it the Cartan matrix of R.

Example 4.4. *Let us compute the Cartan matrix $C(G_2)$.*



Obviously α and β form a base for G_2. We have already computed $n(\alpha, \beta) = -1$ and $n(\beta, \alpha) = -3$ in the previous talk. Therefore we get:

$$C(G_2) := \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

Proposition 4.5. *A reduced root system is determined, up to isomorphism, by its Cartan matrix. More precise: Let R be a reduced root system in V with base S and let R' in V' be also a reduced root system with base S'. Assume, there is a bijection $\phi : S \to S'$, s.t. $n(\phi(\alpha), \phi(\beta)) = n(\alpha, \beta)$ for all $\alpha, \beta \in S$. Then there is a unique isomorphism $f : V \to V'$ which extends ϕ and maps R to R'.*

Proof. Since, S and S' are bases, we can define f uniquely by linear extension of ϕ. Therefore f is an isomorphism and we only need to check, that it maps R to R'. Let $\alpha, \beta \in S$, we have

$$s_{\phi(\alpha)} \circ f(\beta) = s_{\phi(\alpha)}(\phi(\beta)) = \phi(\beta) - n(\phi(\beta), \phi(\alpha))\phi(\alpha) = \phi(\beta) - n(\beta, \alpha)\phi(\alpha)$$

and on the other hand

$$f \circ s_\alpha(\beta) = f(\beta - n(\beta, \alpha)\alpha) = f(\beta) - n(\beta, \alpha)f(\alpha).$$

Therefore we see $s_{\phi(\alpha)} \circ f = f \circ s_\alpha$ for all $\alpha \in S$. As a consequence of Theorem 3.1 we see that $W(R') = f \circ W(R) \circ f^{-1}$. By the same theorem we also know, that $R = W(R)(S)$ and $R' = W(R')(S')$, hence we deduce

$$f(R) = f(W(R)(S)) = W(R')(f(S)) = W(R')(S') = R'.$$

□

5. The Coxeter graph

In the following section we want to introduce a bit of graph theory, in particular the notion of a multi-edge graph and the Coxeter graph.

Definition 5.1. *Let V be a non empty finite set, we call the points in V* ***vertices****. Moreover let $W := \{\{u, v\} \subset V \mid u \neq v\}$ and fix a map $E : W \to \mathbb{N}_0$, counting the* ***edges*** *between different vertices. We call the pair (V, E) a* ***(finite) multi-edge graph****.*

Definition 5.2. *Let $G = (V, E)$ and $G' = (V', E')$ be two finite multi-edge graphs. We call an injective map $p : V \to V'$ a* ***homomorphism****, if $E(\{u, v\}) \leq E'(\{p(u), p(v)\})$ for all $\{u, v\} \in W$.*
We call an injective map $p : V \to V'$ an ***isomomorphism****, if p is bijective and the inverse is also a homomorphism.*

Definition 5.3. *We call a graph* ***connected****, if for all $x, y \in V$ there exists a finite sequence $(v_i)_{i=0,\cdots,n}$, s.t. $v_i \neq v_{i-1}$, $v_0 = x$ and $v_n = y$ and $E(\{v_{i-1}, v_i\}) > 0$ for all $i = 1, \cdots n$.*

Definition 5.4. *A* ***Coxeter graph*** *is a finite multi-edge graph (V, E), s.t. $E(\{u, v\}) \leq 3$ for all $\{u, v\} \in W$.*

Definition 5.5. *Given a root system R with base S we define the Coxeter graph with respect to S as follows: $V = S$ and $E(\{\alpha, \beta\}) = n(\alpha, \beta) \cdot n(\beta, \alpha)$ for all $\alpha \neq \beta \in S$.*

By the following lemma the Coxeter graph of a root system is independent of the choice of S, therefore we call it just the Coxeter graph of R.

Lemma 5.6. *Let R be a root system with bases S and S'. Then there exists an isomorphism between the Coxeter graph of R with respect to S and the Coxeter graph of R with respect to S'.*

Proof. By Theorem 3.1 $W(R)$ acts transitively on the set of bases of R. Therefore there exists a $w \in W(R)$, s.t. $w(S) = S'$. In particular, w is a bijection between S and S'. Since $n(\alpha, \beta) = n(w(\alpha), w(\beta))$, w defines an isomorphism of graphs. □

Example 5.7. *The Coxeter graph of G_2 is given by*

 .

Proposition 5.8. *Suppose $V = V_1 \oplus V_2$ and $R \subset V_1 \cup V_2$. Let $R_i := V_i \cap R$, then the following statements are true:*

1. V_1 *and* V_2 *are orthogonal,*
2. R_i *is a root system in* V_i.

Proof. 1. Since V_i is spanned by R_i, it is sufficient to show $(\alpha, \beta) = 0$ for all $\alpha \in R_1, \beta \in R_2$. Therefore let $\alpha \in R_1, \beta \in R_2$, then $\alpha - \beta \notin V_1 \cup V_2$, if it would, then α or β would be in $V_1 \cap V_2 = \{0\}$. Using Lemma A.1, we have $(\alpha, \beta) \leq 0$. If we apply the same arguments to $\alpha \in R_1$ and $-\beta \in R_2$ we see $(\alpha, \beta) \geq 0$ and therefore $(\alpha, \beta) = 0$.

2. The only non trivial condition we have to check is, that given $\alpha \in R_i$ there is a symmetry $s'_\alpha \in \mathrm{GL}(V_i)$ leaving R_i invariant. We choose $s_\alpha \in \mathrm{GL}(V)$ the symmetry associated to α given by the root system R. By (1.) it follows that $s_{\alpha|V_2} = \mathrm{id}_{V_{3-i}}$. Since $\mathrm{ord}(s_\alpha) = 2$, we have $s_\alpha(V_i) \subset V_i$, in particular $s_\alpha(R_i) = R_i$.

□

Remark 5.9. We call the R_i subsystems of R and say R is the sum of its subsystems.

Definition 5.10. *We call a root system R **irreducible** if the assumption in Proposition 5.8 is only satisfied trivially, i.e.* V_1 *or* V_2 *equals to* 0.

Remark 5.11. It is a obvious observation that each root system is a sum of irreducible root systems. One can also show that this decomposition is unique.

Now we want to express irreducibility regarding to the corresponding Coxeter graph. We have the following

Proposition 5.12. *Let R be a root system. Then R is irreducible if and only if its Coxeter graph is connected and non empty.*

Proof. We assume that R is the sum of two non trivial subsystems R_1 and R_2. If S_1 and S_2 are bases for R_1 and R_2, respectively, and $S = S_1 \cup S_2$, then S is a base for R. If $\alpha \in S_1$ and $\beta \in S_2$, then they are orthogonal by Proposition 5.8 and are therefore not joined by any edge in the Coxeter graph of R. So the Coxeter graph is the disjoint sum of the Coxeter graphs of the R_i and therefore not connected.
Conversely, assume the Coxeter graph of R is not connected, i.e. there is a non trivial partition $S = S_1 \sqcup S_2$, where S is a base of R, where all elements of S_1 are orthogonal to any element in S_2. The corresponding subspaces $V_i = \mathrm{span}\,(S_i)$ are also orthogonal and, as seen in the proof of Proposition 5.8, invariant under the symmetries S_α, for $\alpha \in S$. By Theorem 3.1 for each $\beta \in R$ there is a $w \in W(R) = W_S$, s.t. $w(\beta) \in S \subset V_1 \cup V_2$. Using the invariance we can conclude, that β has to be in $V_1 \cup V_2$. Hence R is in $V_1 \cup V_2$ and therefore reducible. □

6. Dynkin Diagrams and Classification

We've seen in Proposition 4.5 that a root system is uniquely determined by its Cartan matrix (up to isomorphism). If we compute the Coxeter diagramms of B_n and C_n for $n \geq 3$, we get the same graph:

 .

Therefore we see that we need to attach more information to the diagram to uniquely determine the root system. Looking up the definition of the Coxeter graph we see, that we only store the information of the angle between two base roots and not which of them is longer, which is an information contained in the Cartan matrix. This leads us to the following

Definition 6.1. *Let R be a root system and $G = (V, E)$ its Coxeter graph. If we label each vertex with the square of the length of the corresponding root, then this graph is called* ***Dynkin diagram****. Given two Dynkin diagrams (V, E) and (V', E') and a map f between them, f is called a isomorphism, if it is a isomorphism of the underlying Coxeter graphs and there exists $c \in \mathbb{R}_{>0}$, s.t. $(v, v) = c(f(v), f(v))$ for all $v \in V$.*

Example 6.2. *The Dynkin diagram of G_2 is given by*

.

Remark 6.3. Using the previous talk we know that the actual value of the square of the length of the corresponding root is not stored in the Cartan matrix. The matrix only remembers which root has greater length. Moreover, we see that if two roots are connected by only one edge, then they have the same length, so we can replace the label by just adding an arrow pointing to the shorter root. For instance, the Dynkin diagram of G_2 is now given by

 .

Proposition 6.4. *The Dynkin diagram of a root system R determines the Cartan matrix of R.*

Proof. This proof will be given as an algorithm to compute the Cartan matrix from the Dynkin diagram. By Lemma A.2 we know, that given $\alpha \neq \beta \in S$, with S being a base for R, then $(\alpha, \beta) \leq 0$, and therefore $n(\alpha, \beta) \leq 0$. So given $\alpha, \beta \in S$ we have:

- if $\alpha = \beta$ then $n(\alpha, \beta) = 2$,
- if $\alpha \neq \beta$, and if α and β are not joined by an edge, then $n(\alpha, \beta) = 0$,
- if $\alpha \neq \beta$, and if α and β are joined by at least one edge,
  - if $(\alpha, \alpha) \leq (\beta, \beta)$, then $n(\alpha, \beta) = -1$,
  - if $(\alpha, \alpha) \geq (\beta, \beta)$, then $n(\alpha, \beta) = -\#$ of joined edges.

These are all direct consequences of the table of possible values for $n(\alpha, \beta)$ from the previous talk. □

Corollary 6.5. *A reduced root system is determined, up to isomorphism, by its Dynkin diagram.*

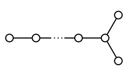
Using Proposition 5.12 and the previous Corollary 6.5 we see that it is enough to classify the connected Dynkin diagrams, more precisely one can show the following

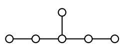
Theorem 6.6. *If R is an irreducible root system of rank n, then its Dynkin diagram is one of the following:*

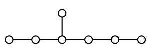
A_n $(n \geq 1)$: ,

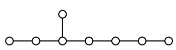
B_n $(n \geq 2)$: ,

C_n $(n \geq 3)$: ,

D_n $(n \geq 4)$:  ,

E_6 :  ,

E_7 :  ,

E_8 :  ,

F_4 : ,

G_2 : .

Proof. See [Hum72] 11.4. □

7. Construction of irreducible root systems

Theorem 6.6 gives us the classification we wanted. One can further ask if we are able to not only classify but identify simple Lie algebras by their Dynkin diagrams. To do this we need to show, that each Dynkin diagram from Theorem 6.6 actually correspond to a irreducible root system. We do this by explicitly constructing the root systems. Moreover, one has to show, that for each such root system there exists a simple Lie algebra, s.t. its root system decomposition produces this root system and that the choice of the Cartan subalgebra doesn't change the root system (up to isomorphism). These are proven in [Hum72] 18.3 and 16.4., respectively.

Proposition 7.1. *Each Dynkin diagram appearing in Theorem 6.6 can be constructed as a Dynkin diagram of an irreducible root system.*

Proof. Let $e_1, \cdots, e_n$ be the standard basis of $\mathbb{R}^n$. We endow $\mathbb{R}^n$ with the standard inner product $(-,-)$ and define L_n to be the complete lattice spanned by the standard basis. In all cases the irreducibility follows by Proposition 5.12 after constructing the Dynkin diagram.

- A_n $(n \geq 1)$: We take $V := (e_1 + \cdots + e_{n+1})^\perp \subset \mathbb{R}^{n+1}$ and

$$R = \{\alpha \in V \cap L_{n+1} \mid (\alpha, \alpha) = 2\} = \{e_i - e_j \mid i \neq j\}.$$

The symmetry s_α associated to $\alpha \in R$ can be written as

$$\beta \mapsto \beta - (\alpha, \beta)\alpha.$$

It is now obvious, that R is a reduced root system. If we take the subset $S := \{e_i - e_{i+1} \mid 1 \leq i \leq n\}$, then we have for $i < j$ with a telescope sum

$$e_i - e_j = e_i - e_{i+1} + \cdots + e_{j-1} - e_j$$

which shows, that S is a base for R since the elements in S span V. Moreover, we see $(e_i - e_{i+1}, e_j - e_{j+1}) \neq 0$ iff $|i - j| = 1$ or $i = j$. If $|i - j| = 1$, the inner product can be computed to -1 and therefore the Dynkin diagram is given by

 .

- B_n $(n \geq 2)$: We take $V := \mathbb{R}^n$ and $R := \{\alpha \in L_n \mid (\alpha, \alpha) = 1 \text{ or } (\alpha, \alpha) = 2\} = \{\pm e_i \mid i = 1, \cdots, n\} \cup \{\pm e_i \pm e_j \mid 1 \leq i \neq j \leq n\}$. A symmetry associated to $\alpha \in R$ can be written as

$$\beta \mapsto \beta - (\alpha, \beta)\alpha \text{ or } \beta \mapsto \beta - 2(\alpha, \beta)\alpha.$$

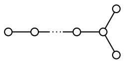
Hence it is again obvious that R is a root system. Now again using telescope sums we see that $S = \{e_i - e_{i+1} \mid 1 \leq i \leq n-1\} \cup \{e_n\}$ is a base for R. Moreover, we see $(e_i - e_{i+1}, e_n) \neq 0$ iff $i = n - 1$ and in this case the inner product equals to -1 and hence $n(\alpha, \beta)n(\beta, \alpha) = -2$, therefore the Dynkin diagram is given by

 .

- C_n $(n \geq 3)$: We choose the dual root system of B_n. Hence $V = \mathbb{R}^n$ and $R = \{\pm 2e_i \mid i = 1, \cdots, n\} \cup \{\pm e_i \pm e_j \mid 1 \leq i \neq j \leq n\}$ is a root system with base $S = \{e_i - e_{i+1} \mid 1 \leq i \leq n-1\} \cup \{2e_n\}$ (this is a result of the previous talk). Computing the Cartan integers we see that the Dynkin diagram is given by

 .

- D_n $(n \geq 4)$: Let $V := \mathbb{R}^n$ and $R := \{\alpha \in L_n \mid (\alpha, \alpha) = 2\} = \{\pm e_i \pm e_j \mid 1 \leq i \neq j \leq n\}$. A symmetry can be written as
$$\beta \mapsto \beta - (\alpha, \beta)\alpha.$$
Hence it is obvious, that R is a root system, with telescope sums we see that a base of R is given by $S = \{e_i - e_{i+1} \mid 1 \leq i \leq n-1\} \cup \{e_{n-1} + e_n\}$. Computing the Cartan integers we get the Dynkin diagram

 .

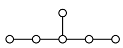
- G_2: Let $V := (e_1 + e_2 + e_3)^\perp \subset \mathbb{R}^3$ and
$$\begin{aligned} R &= \{\alpha \in V \cap L_3 \mid (\alpha, \alpha) = 2 \text{ or } (\alpha, \alpha) = 6\} \\ &= \pm\{e_1 - e_2, e_2 - e_3, e_3 - e_1, 2e_1 - e_2 - e_3, e_1 - 2e_2 - e_3, e_1 - e_2 - 2e_3\}. \end{aligned}$$
As a base we can choose $S = \{e_1 - e_2, -2e_1 + e_2 + e_3\}$ and compute the Dynkin diagram

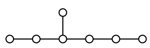
 .

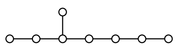
- F_4: Let $V = \mathbb{R}^4$ and $L = L_4 + \frac{1}{2}(e_1 + e_2 + e_3 + e_4)\mathbb{Z}$, then $R := \{\alpha \in L \mid (\alpha, \alpha) = 1 \text{ or } 2\}$. Then
$$R = \{\pm e_i \mid 1 \leq i \leq 4\} \cup \{\pm e_i \pm e_j \mid 1 \leq i \neq j \leq 4\} \cup \{\pm\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\}.$$
A easy verification shows, that the Cartan numbers are actually integers and therefore, that R is a root system. One can choose a base to be $S = \{e_2 - e_3, e_3 - e_4, e_4, \frac{1}{2}(e_1 - e_2 - e_3 - e_4)\}$ and hence the Dynkin diagram is given by

 .

- E_6, E_7, E_8: After constructing E_8 we construct E_6 resp. E_7 by taking the intersection of E_8 with the subspace spanned by the first 6 resp. 7 base vectors. For E_8 let $V = \mathbb{R}^8$ and $L = L_8 + \frac{1}{2}(e_1 + \cdots + e_8)\mathbb{Z}$ and define $L' := \{\sum_{i=1}^8 a_i e_i + \frac{a}{2}(e_1 + \cdots + e_8) \in L \mid \sum_{i=1}^8 a_i + a \in 2\mathbb{Z}\}$. This is again a lattice in $\mathbb{R}^8$. We take $R = \{\alpha \in L' \mid (\alpha, \alpha) = 2\}$, then one can compute R to be $\{\pm e_i \pm e_j \mid 1 \leq i \neq j \leq 8\} \cup \{\frac{1}{2}\sum_{i=1}^8 (-1)^{a_i} e_i \mid \sum_{i=1}^8 a_i \in 2\mathbb{Z}\}$. Computing the Cartan numbers one sees that R is a root system. As a base one can choose $S = \{\frac{1}{2}(e_1 - (e_2 + \cdots + e_7) + e_8), e_1 + e_2, e_2 - e_1, e_3 - e_2, e_4 - e_3, e_5 - e_4, e_6 - e_5, e_7 - e_6\}$. One gets the Dynkin diagrams

E_6 :  ,

E_7 :  ,

E_8 :  .

□

A. Appendix

In this appendix we state a few statements about root systems and bases for root systems and where you can find a proof for them. In this section R will always be a root system in V.

Lemma A.1. *Let α, β be non proportional roots. Assume $(\alpha, \beta) > 0$, then $\alpha - \beta$ is a root.*

Proof. See [Hum72] Lemma in 9.4 or [Ser00] Proposition 3 in V.7. ☐

Lemma A.2. *Given $\alpha \neq \beta \in S$ for a base S for R. Then $(\alpha, \beta) \leq 0$.*

Proof. See [Hum72] Lemma in 10.1 or [Ser00] Lemma 3 in V.8. ☐

Definition A.3. *Let $t^* \in V^*$ be an element s.t. $t^*(\alpha) \neq 0$ for all $\alpha \in R$. We define $R^+_{t^*} := \{\alpha \in R \mid t^*(\alpha) > 0\}$. Then we call an element of $R^+_{t^*}$ **decomposable**, w.r.t. t^*, if there exists $\beta, \gamma \in R^+_{t^*}$, s.t. $\alpha = \beta + \gamma$, otherwise α is called **indecomposable**. We define $S_{t^*} := \{\alpha \in R^+_{t^*} \mid \alpha \text{ is indecomposable}\}$.*

We have the following

Theorem A.4. *S_{t^*} is a base for R. In particular there always exists a base. Moreover, given a base S for R and an element $t^* \in V^*$, s.t. $t^*(\alpha) > 0$ for all $\alpha \in S$ then $S = S_{t^*}$.*

Proof. See [Hum72] 10.1 or [Ser00] V.8. ☐

Lemma A.5. *Let S be a base, and R_+ as in the recap section. Let $\rho := \frac{1}{2} \sum_{\alpha \in R^+} \alpha$, then*

$$s_\alpha(\rho) = \rho - \alpha \text{ for all } \alpha \in S. \tag{1}$$

Proof. See the Corollary of Lemma B in [Hum72] 10.2 or the Corollary of Proposition 6 in [Ser00] V.9. ☐

References

[Hum72] J.E. Humphreys. *Introduction to Lie Algebras and Representation Theory.* Graduate texts in mathematics. Springer, 1972. ISBN: 9780387900537. URL: https://books.google.de/books?id=TiUlAQAAIAAJ.

[Ser00] J.P. Serre. *Complex Semisimple Lie Algebras.* Springer Monographs in Mathematics. Springer Berlin Heidelberg, 2000. ISBN: 9783540678274. URL: https://books.google.de/books?id=7AHsSUrooSsC.