Talk 5 - Root systems and root space decomposition

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5.1 Root space decomposition

From here on out we shall consider L to be a non-zero semisimple Lie algebra and F to be an algebraicly closed field. As we have already seen, this is equivalent to $[L, L] = L$ or its center being zero. Also keep in mind the example of $L = \mathfrak{sl}(2, F)$ (or $L = \mathfrak{sl}(n, F)$ for $2 \leq n \in \mathbb{N}$) which will be helpful to get a clearer image of the following definitions and concepts.

For L being semisimple we are able to find an element $x \in L$ which has a nonzero semisimple part x_s of its abstract *Jordan decomposition*. Thus we know that L has a nonzero semisimple subalgebra (i.e. the span of x_s) which only consists of semisimple elements. We call such a subalgebra **toral**.

Lemma 1. *A toral subalgebra of L is abelian.*

Proof. Let $H \subset L$ be a toral subalgebra. We have to show that $ad_H H = 0$ for all elements in H. For $ad\ x$ being semisimple and F being algebraically closed we know that $ad\ x$ is diagonalizable. Thus we just need to show that it has no nonzero eigenvalues. Supposing the opposite one will reach a contradiction rather fast. □

Now we want to fix a maximal toral subalgebra $H \subset L$. In the case of $L = \mathfrak{sl}(n, F)$ it is easy to see, that such an H consists of all the diagonal matrices (with trace 0). For H being toral wen know that it is abelian and therefore $ad_L H$ is a commuting family of semisimple endomorphisms of L. Using some standard results of linear algebra we can see that $ad_L H$ is simultaneously diagonalizable, thus L can be written as a direct sum of the subspaces $L_\alpha = \{x \in L | [hx] = \alpha(x)h \ for\ all\ h\ in\ H\}$ where α ranges over H^*. For $\alpha = 0$ we can view L_0 simply as $C_L(H)$, the centralizer of H. Further let us denote Φ as the set of all $\alpha \neq 0$ of H^*, such that L_α is not zero. The elements of Φ are refered to as **roots** of L relativ to H. Using this notation we have a **root space decomposition** $(*)$ $L = L_0 \bigoplus \bigsqcup_{\alpha \in \Phi} L_\alpha$ (also called **Cartan decomposition**).

For the next part we want to take a closer look at this decomposition:

Proposition 2. For all $\alpha, \beta \in H^*$ we get $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$. If $x \in L_\alpha$ with $\alpha \neq 0$ then it follows that $ad\ x$ is nilpotent. If α and β in H^* and $\alpha + \beta \neq 0$, then L_α and L_β are orthogonal to each other relativ to the *Killing form* κ_L of L.

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Proof. Using the Jacobi identity for $x \in L_\alpha$, $y \in L_\beta$ and $h \in H$ we get:

$$ad\ h([xy]) = [[hx]y] + [h[xy]] = \alpha(h)[xy] + \beta(h)[xy] = \alpha + \beta(h)[xy]$$

From this we get the first assertion and the second is an immediate consequence of the first. (Why?) The last one can be obtained by using the $Killing\ form$, its compatibility with $[-,-]$ and a chain of equalities which force the $Killing\ form$ to be zero. □

Corollary 3. *The restriction of κ_L to $L_0 = C_L(H)$ is non-degenerate.*

Proof. We already now from previous talks that κ_L is non-degenerate because L is semisimple. Furthermore we have just seen that $L_0 \perp_{\kappa_L} L_\alpha$ for all $\alpha \in \Phi$. Now if there is a $z_0 \in L_0$ it would follow that $\kappa(z_0, L) = 0$ which would force z_0 to be zero. □

Using a fact from linear algebra and after learning the facts above we are ready to proof the equality of a maximal toral subalgebra and its centralizer.

Lemma 4. *If x and y are commuting endomorphisms of a finite dimensional vector space, with y being nilpotent, it follows that xy is nilpotent as well. In particular we get $Tr(xy) = 0$.*

Proposition 5. Let H be a maximal toral subalgebra of L. Then it follows that $H = C_L(H) =: C$.

Proof. This will only be a scatch of the proof, but it's a good excercise to do it more in detail yourself.

(1) C contains the semisimple and nilpotent parts of all its elements ($ad\ x(H) = 0$ for all $x \in C$)

(2) all semisimple elements of C lie in H (H is maximal)

(3) κ_H is non-degenerate (straight calculation)

(4) C is nilpotent (Engel's Thm. + lin. alg. fact)

(5) $H \cap [C, C]$ (κ's compatibility with the bracket)

(6) C is abelian (assume $[C, C] \neq 0$)

(7) $C = H$ (assume that's not the case)

□

5.2 Othogonality properties

As a dircet consequence we get the fact that κ_H is non-degenerate and this allows us now to identify H with H^* as follows:
To an $\alpha \in H^*$ corresponds a (unique) element $t_\alpha \in H$ whiuch satisfies $\alpha(h) = \kappa(t_\alpha, h)$ for all $h \in H$. In particular, Φ corresponds to $\{t_\alpha \mid \alpha \in \Phi\}$, which is a subset of H.

Coming back to $L = \mathfrak{sl}(n, F)$, we can observe that $(*)$ corresponds to the decomposition of L given by its standard basis. Let $n = 2$, then the basis would be given by $\{x = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right),\ y = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right),\ h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\}$. As we will see later on, we can identify L_α with $\langle x \rangle$, $L_{-\alpha}$ with $\langle y \rangle$ and $L_0 = C_L(H) = H$ with $\langle h \rangle$. Thus the set of roots of L is given in this case by some α and $-\alpha$ which correspond to $t_\alpha = \left(\begin{smallmatrix} 0.25 & 0 \\ 0 & -0.25 \end{smallmatrix}\right)$ and $t_{-\alpha} = \left(\begin{smallmatrix} -0.25 & 0 \\ 0 & 0.25 \end{smallmatrix}\right)$.

Our next step will be to take a closer look at some more (orthogonality) properties of the L_α.

Proposition 6.

(a) The set of all roots Φ spans H^*

(b) If $\alpha \in \Phi$ then $-\alpha \in \Phi$

(c) If $\alpha \in \Phi$, $x \in L_\alpha$ and $y \in L_{-\alpha}$ then we get $[x, y] = \kappa(x, y)t_\alpha$.

(d) If $\alpha \in \Phi$ then $[L_\alpha.L_{-\alpha}]$ is one dimensional and is spanned by t_α

(e) $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$ for all $\alpha \in \Phi$.

(f) If $\alpha \in \Phi$ and $0 \neq x_\alpha$ is an element of L_α, then there exists an y_α of $L_{-\alpha}$ such that x_α, y_α and $h_\alpha := [x_\alpha, y_\alpha]$ span a 3-dimensional subalgebra $S \cong \mathfrak{sl}(2, F)$ via $x_\alpha \mapsto \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $y_\alpha \mapsto \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $h_\alpha \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

(g) $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$ and also we get $h_\alpha = -h_{-\alpha}$

Proof. to (a): Assuming the opposite we can find (by duality) and nonzero element $h \in H$ such that $\alpha(h) = 0$ for all $\alpha \in \Phi$. This also means that $[h, L_\alpha] = 0$ for all $\alpha \in \Phi$ and for H being abelian $[h, H] = 0$ als holds. Thus it follows that $h \in Z(L)$ which is absurd. (bc. $h \neq 0$ but $Z(L) = 0$ for L being semisimple)

to (b): Let $\alpha \in \Phi$ and assume that $-\alpha \notin \Phi$. We already have seen that $\kappa_L(L_\alpha, L_\beta) = 0$ for all $\beta \in \Phi$ such that $\alpha + \beta \neq 0$ which in this case holds for all those β. Thus it follows that $\kappa_L(L_\alpha, L) = 0$ which condraticts the nondegeneracy of κ_L.

to (c): Let $\alpha \in \Phi$, $x_\alpha \in L_\alpha$ and $y\alpha \in L_{-\alpha}$. Let $h \in H$ be arbitrary. Now we can calculate:

$$\begin{aligned} \kappa(h, [x, y]) &= \kappa([hx], y) = \alpha(h)\kappa(x, y) \\ &= \kappa(t_\alpha, h)\kappa(x, y) = \kappa(\kappa(x, y)t_\alpha, h) = \kappa(h, \kappa(x, y)t_\alpha) \end{aligned}$$

From this we can deduct that H is orthogonal to $[x,y] - \kappa(x,y)t_\alpha$ which forces the euality $[x,y] = \kappa(x,y)t_\alpha$

to (d): As we have seen in (c), t_α spans $[L_\alpha, L_{-\alpha}]$ for $[L_\alpha, L_{-\alpha}]$ not being zero. Now consider $0 \neq x \in L_\alpha$. Then there exists an $0 \neq y \in L_{-\alpha}$, otherwise $\kappa_L(x, L_{-\alpha}) = 0$ wich forces $\kappa_L(x, L)$ to be zero as well. But that contradicts the nondegeneracy of κ_L. From that, by using (c) again, we get that $[x,y] \neq 0$.

to (e): Assuming $\alpha(t_\alpha) = 0$, so that $[t_\alpha, x] = 0 = [t\alpha.y]$ for all $x \in L_\alpha$ and $y \in L_{-\alpha}$. Now we can find x and y (like in (d)) such that $\kappa(x,y) \neq 0$. We may also scale one or the other to get $\kappa(x,y) = 1$. Using (c) we get $[x,y] = t_\alpha$. It follows that the subspace S of L spanned by x, y and t_α is a three dimensional solvable algebra with $S \cong ad_L\ S \subset \mathfrak{gl}(L)$. Furthermore we know that $ad_L\ s$ is nilpotent for all $s \in [S,S]$. Thus $ad_L\ t_\alpha$ is both semisimple and nilpotent. This leads to t_α being an elemnt of $Z(L)$ which contradicts the choice of t_α.

to(f): We want to find an $y_\alpha \in L_{-\alpha}$ for a nonzero $x_\alpha \in L_\alpha$ such that $\kappa(x_\alpha, y_\alpha) = \frac{2}{\kappa(t_\alpha, t_\alpha)}$. This is indeed possible for x_α not being orthogonal to the elements of $L_{-\alpha}$. Now set $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$. One can now verify via easy calculations that the wanted properties hold for h_α. This gives us a three dimensional subalgebra of L which has the same multiplication table as $\mathfrak{sl}(2, F)$.

to (g): Recall the definition of t_α by $\kappa(t_\alpha, h) = \alpha(h)$ for all $h \in H$. This already shows that $t_\alpha = -t_{-\alpha}$. Using the definition of h_α the last assertion follows right thereafter. □

The next two subsections shall only be discussed briefly.

5.3 Integrality properties

For an $\alpha \in \Phi$ we also have $-\alpha \in \Phi$ as we have seen. Like in Prop. 6 (f), we define S_α to be such a subalgebra of L. With the help of the Weyl theorem and some knowledge about the classification of $\mathfrak{sl}(2, F)$-modules we have a complete description of all (finite dimensional) S_α-modules. In particular we can describe $ad_L\ S$. To sum everything we have gathered so far we have the following Proposition:

Proposition 7.

(a) For $\alpha \in \Phi$ we have $dim\ L_\alpha = 1$. In particular $S_\alpha = L_\alpha + L_{-\alpha} + H_\alpha$ for $H_\alpha = [L_\alpha.L_{-\alpha}]$. Furthermore for a given $0 \neq x_\alpha \in L_\alpha$ we can find a $y_\alpha \in L_{-\alpha}$ sucht that $[x_\alpha, y_\alpha] = h_\alpha$.

(b) If $\alpha \in \Phi$ the only multiples of it in Φ are α and $-\alpha$.

(c) If $\alpha, \beta \in \Phi$, then $\beta(h_\alpha) \in \mathbb{Z}$ and $\beta - \beta(h)\alpha \in \Phi$. We call these numbers $\beta(h)$ *Cartan numbers*.

(d) If $\alpha, \beta, \alpha + \beta \in \Phi$ then $[L_\alpha, L_\beta] = L_{\alpha+\beta}$.

(e) Let $\alpha, \beta \in \Phi$ such that $\beta \neq \pm\alpha$. Let r, q be (respectively) the largest integers for which $\beta - r\alpha$ and $\beta + q\alpha$ are roots. Then all $\beta + i\alpha$ are roots for $-r \leq i \leq q$ and $\beta(h_\alpha) = r - q$.

(f) L is generated (as a *Lie algebra*) by the root spaces L_α.

The chapter about the rationality properties is about finding a $\mathbb{Q} -$ *subspace* $E_\mathbb{Q}$ of H^* with the same dimension (with respect to F) as H^* and which can be extended to a real vector space $E := \mathbb{R} \otimes_\mathbb{Q} E_\mathbb{Q}$. By defining $(\gamma.\delta) := \kappa(t_\gamma, t_\delta)$ for all $\gamma, \delta \in H^*$, E is even an euclidian space and Φ contains a basis of E. With this construction we get a $1:1$ correspondence between the pairs (L, H) and (Φ, E). This leads us to the next section about the *root system*. For more detailed information about the last two topics take a look at the sections (8.4) & (8.5) of "Introduction to Lie Algebras and Representation Theory" by Humphreys.

5.4 Root sytems

Befor we learn something about the so called root system we have to do a little detour and focus on reflections. For the rest of this chapter we therefore fix a euclidian space E, i.e. finite dimensional vector space over $\mathbb{R}$ endowed with a positiv definite symmetric bilinear form $(-,-)$. Geometrically speaking we can understand a reflection on E as an invertible linear transformation leaving pointwise fixed some hyperplane (a subspace of codimension one) and sending any vector orthogonal to that plane into its negative. Thus any nonzero vector α determines a reflection σ_α with a reflecting hyperplane $P_\alpha = \{\beta \in E \mid (\beta, \alpha) = 0\}$. Note that any vector $v \in \langle\alpha\rangle$ will determine the same reflecting hyperplane. We can define σ_α more expplicitly as follows:

$$\sigma_\alpha(\beta) = \beta - \tfrac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$$

One can easily verify that the wanted properties for σ_α hold. For $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ occuring more often we want to abbreviate it by writting $\langle\beta, \alpha\rangle$. The following lemma will be useful later on:

Lemma 8. *Let Φ be a finite set which spans E and suppose that all reflections σ_α with $\alpha \in \Phi$ leave Φ invariant. If there is a $\sigma \in GL(E)$ leaving Φ invariant, fixing a pointwise a hyperplane P of E and sending some nonzero $\alpha \in \Phi$ into its negative, then it follows that $\sigma = \sigma_\alpha$ and $P = P_\alpha$.*

Proof. First we define $\tau = \sigma\sigma_\alpha$. From this definition we can see that τ leaves Φ invariant and acts as an identity on $\mathbb{R}\alpha$ as well as on the quotient $E/\mathbb{R}\alpha$. Thus all eigenvalues of τ are one and the minimal polynomial of τ devides $(T-1)^l$ where $l = dim\ E$. On the other hand, since Φ is finte, not all vectors $\beta, \tau(\beta), \ldots, \tau(\beta)^k$ can be distinct ($\beta \in \Phi$ and $|\Phi| \leq k$). As a consequence of that there exists a power of τ which fixes β. Now chose k large enough that τ^k fixes all $\beta \in \Phi$. Because Φ spans E this forces $\tau^k = 1$. Thus the minimal polynomial of τ devides $T^k - 1$. Combined with the previous observation this shows that the minimal polynomial of τ equals $T-1 = gcd\{T^k-1,(T-1)^l\}$ which means that $\tau = 1$. □

Having this in mind we can go on to our desired definition of the root system:

Definition 9. A subset Φ of an euclidian space E is called **root system** in E if the following axioms are satisfied:
$(R1)$ Φ is finte, spans E and does not contain 0
$(R2)$ If $\alpha \in \Phi$ then the only multiples of it in Φ are α and $-\alpha$
$(R3)$ If $\alpha \in \Phi$, the reflection σ_α leaves Φ invariant.
$(R4)$ If $\alpha, \beta \in \Phi$ then $\langle \beta, \alpha \rangle \in \mathbb{Z}$

In some literature one may find this definition without the inclusion of $(R2)$. What we call a "root system"here might there be refered to as a "reduced root system". Also note that $(R2)$ and $(R3)$ imply that $\Phi = -\Phi$.

Let Φ be a root system in E and denote by $\mathcal{W}$ the subgroup of $GL(E)$ generated by the reflections σ_α for α being an element of Φ. By $(R3)$, $\mathcal{W}$ permutes the set Φ, which is finte because of $(R1)$. This allows us to identify $\mathcal{W}$ with a subgroup of the symmetric group S_n for $n := |\Phi|$. We call this group the **Weyl group** of Φ. In the following lemma we will see how certain automorphisms of E act on $\mathcal{W}$.

Lemma 10. *Let Φ be a root system in E with the Weyl group $\mathcal{W}$. If $\sigma \in GL(E)$ leaves Φ invariant, then $\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$ for all $\alpha \in \Phi$ and $\langle \beta, \alpha \rangle = \langle \sigma(\beta), \sigma(\alpha) \rangle$ for all $\alpha, \beta \in \Phi$.*

The Weyl group and its properties will be discussed more in detail in the next talk.

References

[Hum72] James Humphreys. Introduction to Lie Algebras and Representation Theory. Springer, 1972. url: https://www.math.uci.edu/ brusso/humphreys.pdf.