Representations of the Lie Algebra $\mathfrak{sl}_n(\mathbb{C})$

Aljoscha Helm

15th of December 2023

Introduction

These are the notes corresponding to the fourth talk given during a seminar on semi-simple Lie algebras at Heidelberg University under the supervision of Professor Florent Schaffhauser. They are split into two main sections: The representations of $\mathfrak{sl}_2(\mathbb{C})$ and those of $\mathfrak{sl}_n(\mathbb{C})$. The first section contains everything that was said during the presentation, and should serve as a very thorough example of the second, more general case, as the concepts stay similar, but the terminology and methods used get more abstract. The definitions in the first chapter are to be treated as preliminary, since they suffice in the way they are given, when one is studying $\mathfrak{sl}_2$, but will need to be changed, or generalised when moving to $\mathfrak{sl}_n$. In the sense of a certain brevity there are a few "inaccuracies" in the second section as well, but these are usually marked as such. Especially the *universal enveloping algebra*, mentioned in section 2.4 is affected by this and I recommend using either [Ser92] or [Hal15] as references for filling those gaps.

If not explicitly defined otherwise, $\mathfrak{sl}_n$ will denote $\mathfrak{sl}_n(\mathbb{C})$ and all $\mathfrak{sl}_n$-modules are assumed to be finite dimensional.

1 Representations of $\mathfrak{sl}_2(\mathbb{C})$

Recall

Let V be a vector space, $X, Y \in \mathfrak{g}$.

1. A representation of a Lie algebra is a linear map:

$$\pi : \mathfrak{g} \to \mathrm{End}(V) \quad \text{satisfying the relation} \quad \pi([X,Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X).$$

2. Let $\cdot_\pi : \mathfrak{g} \times V \to V, (X, v) \mapsto \pi(X)(v)$. Then the pair $(V, \cdot_\pi)$ is called a $\mathfrak{g}$-module.

By abuse of terminology one usually writes "V is a $\mathfrak{g}$-module" and $\pi(X)(v) = X \cdot v = Xv$ are used in an equivalent way. In some references the linear map, which effectively defines the representation is only implied or V itself is even referred to as the representation, but as it is possible to identify endomorphisms with matrices and they are not studied explicitly in many cases this small inaccuracy is not uncommon. Especially when studying well-understood Lie-algebras like $\mathfrak{sl}_2$ or $\mathfrak{sl}_n$ this does not pose a problem, since the representations are essentially identified by the eigenvalues or rather weights of H and $\mathfrak{h}$ respectively.

The canonical basis for $\mathfrak{sl}_2(\mathbb{C})$ is given by:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

satisfying the bracket relations: $[X,Y] = H \qquad [H,X] = 2X \qquad [H,Y] = -2Y$.

The choice of the given elements arises naturally but, as we will see later on, there is a much more methodical approach to defining the basis of $\mathfrak{sl}_n$ for any n.

1.1 Weights and Primitive Elements of $\mathfrak{sl}_2$

Definition 1.1.1
Let V be a finite dimensional $\mathfrak{sl}_2$-module and $\lambda \in \mathbb{C}$, an eigenvalue to H.

V_λ denotes the eigenspace of H in V, corresponding to λ, i.e.: $V_\lambda = \{v \in V \mid Hv = \lambda v\}$, which is called *weightspace* in this context.

Elements $v \in V_\lambda$ are said to have *weight* λ whereas λ itself is called a *weight* of V

$e \in V \backslash \{0\}$ is called *primitive element* of weight λ if (and only if) it is an eigenvector to H with weight λ and is terminated by applying X to it, i.e.: $Xe = 0$ and $He = \lambda e$

Proposition 1.1.2.

(i) The vectorspace V is a direct sum of weight spaces:

$$\bigoplus_{\lambda \in \mathbb{C}} V_\lambda = V.$$

(ii) If v is an element of V_λ, hence has weight λ, then the elements Xv and Yv have weight $\lambda + 2$ and $\lambda - 2$ respectively.

(iii) Every non-empty, finite dimensional $\mathfrak{sl}_2$-module contains a primitive element.

Proof.

(i) Since $\mathbb{C}$ is algebraically closed all eigenvalues of H are elements of $\mathbb{C}$ and they are distinct, hence V is a direct sum of (eigen-/) weightspaces of H.

(ii) This part can be calculated directly. Let $v \in V_\lambda$, then:

$$HXv = [H, X]v + XHv = 2Xv + X\lambda v = (\lambda + 2)Xv \implies Xv \in V_{\lambda+2}.$$

Here we used the basic bracket relations mentioned earlier and the fact that v was chosen as an eigenvector to H.

(iii) Let v again be an eigenvector to H and consider the sequence $v, Xv, X^2v, \ldots.$ This sequence terminates, as V is finite dimensional, hence there will be a m such that $V_{\lambda+2m} \neq 0$ but $V_{\lambda+2m+1} = 0$. The last non-zero element of this sequence $X^m v$ will be the proposed primitive element.

□

1.2 $\mathfrak{sl}_2$-Modules Generated by Primitive Elements

Due to Prop.1.1.2(ii) we may observe, that by applying X and Y to elements of some weightspace we are able to raise or lower their respective weight. This is the first instance, in which we can (albeit still rather heuristically) observe that if any module contains weights with respect to H and is stable under the action of X, Y, then it must contain elements of every possible weight. Moreover, as V is assumed to be finite dimensional, there must be a weightspace consisting of elements whose weight can not be raised any further, if we apply Y to an element of this space of *highest weight* we can span all of V. This is one of the goals of this section.

$$\begin{tikzcd}
V_{\lambda-4} \arrow[r, bend left, "X"] & V_{\lambda-2} \arrow[l, "Y"] \arrow[r, bend left, "X"] & V_\lambda \arrow[l, "Y"]
\end{tikzcd}$$

Proposition 1.2.1. *Let e be a primitive element of weight λ and $e_n = Y^n e \frac{1}{n!}$ with the convention that $e_{-1} = 0$. Then the following three formulas hold:*

(i) $He_n = (\lambda - 2n)e_n$

(ii) $Ye_n = (n + 1)e_{n+1}$

(iii) $Xe_n = (\lambda - n + 1)e_{n-1}$

Proof. For this proof please refer to Section 5 Lemma 2.5.2. □

These formulas are a formal version of the prior observation that Y can be used to span V, but we can also draw a few important corollaries from this proposition.

Corollary 1.2.2. *There is an integer m, such that $e_i = 0\ \forall i > m$, the eigenvectors $e_1, \dots, e_i$ are linearly independent, and the corresponding eigenvalues are integers.*

Proof. The eigenvectors are linearly independent, due to them having distinct weights. Furthermore, as V is finite dimensional there must exist an $m \in \mathbb{N}$ such that $V_{\lambda+m} \neq 0$, but $V_{\lambda+(m+1)} = 0$, hence $e_i = 0 \forall i > m$ and, as applying formula (iii) to the previous statement shows, $\lambda = m \in \mathbb{N}$. This m is then called *highest weight* of V. □

Our goal was to generate $\mathfrak{sl}_2$-modules from primitive elements, this can be achieved if we consider the submodule $W \subseteq V$ with a basis given by $B_W = \{e, \dots, e_m\}$.

Corollary 1.2.3.

(i) W is stable under $\mathfrak{sl}_2$.

(ii) W is an irreducible $\mathfrak{sl}_2$-module.

Proof.

(i) The formulas show $H(W), X(W), Y(W) \subseteq W$.

(ii) Let $W' \subseteq W$, non-zero and stable under $\mathfrak{sl}_2$. The eigenvalues of H in W are given by $m, m-2, m-4, \dots, -m$, each with multiplicity 1. As W' is defined to be a non-zero subspace of W and is assumed to be stable under $\mathfrak{sl}_2$, it has to contain one of the eigenvectors e_i. By applying formulae (ii) and (iii) we can then lower or raise the weight of this e_i, such that we reach $e_0, \dots, e_{i-1}, e_i, e_{i+1}, \dots, e_m$. This proves $W' = W$ and W is irreducible.

□

1.3 Classifying $\mathfrak{sl}_2$-Modules by Weight

We will now consider a more general case, in which the action of $\mathfrak{sl}_2$ is not necessarily given by elements of the canonical basis, but any endomorphisms[1] satisfying the following conditions:

Let W_m be a vectorspace with a basis $\mathcal{B}_m = \{e_0, \dots, e_m\}$ and thereby $\dim W_m = m + 1$ and let h, x, y be endomorphisms on W_m. If the following formulas hold, h, x, y turn W_m into a $\mathfrak{sl}_2$-module, as seen in the previous section:

$$\begin{aligned} &he_n = (m-2)e_n, && ye_n = (n+1)e_{n+1}, && xe_n = (m-n+1)e_{n-1}, \\ &hxe_n - xhe_n = 2xe_n, && hye_n - yhe_n = -2ye_n, && xye_n - yxe_n = he_n. \end{aligned}$$

Theorem 1.3.1. *Define W_m as above and let V be an irreducible $\mathfrak{sl}_2$-module of dimension $m+1$, then we get:*

(i) W_m is irreducible.

(ii) $V \cong W_m$.

Proof.

(i) This follows from 1.2.3 and the fact that W_m is generated by images of e_0 with weight m in a similar way to Y spanning V in the last section.

(ii) We already know that V contains a primitive element v of integer weight w, and that the submodule V' of V generated by this v has dimension $w+1$. As V was presumed to be irreducible we can infer that $V' = V$ and $w = m$, which then provides us with the fact that $V \cong W_m$, as we can simply apply the formulas we defined earlier.

□

This is one of, if not *the* most important part of this topic, as we have now proven, that the irreducible representations of $\mathfrak{sl}_2$ are in a $1:1$-correspondence to a system of integers and are as such defined by their highest weight.

[1] We will assume these endomorphisms and representations of dimension $m+1$ to exist, as this proof would need additional knowledge of Lie group representation, for details cf. [Hal15]Ch. 4.2

Theorem 1.3.2. *A finite dimensional $\mathfrak{sl}_2$-module V is isomorphic to a direct sum of W_m-modules.*

Proof. Due to *Weyl's Theorem* every finite dimensional linear representation of semi-simple Lie algebras is completely irreducible. From this the theorem can by directly deduced. $\square$

Theorem 1.3.3.

(i) The induced endomorphism of V is diagonalizable with integer eigenvalues and for any eigenvalue n, the elements $n-2, n-4, \dots, -n$ are also eigenvalues.

(ii) Y and X induce isomorphisms:

$$X^n : V_n \to V_{-n} Y^n : V_{-n} \to V_n$$

Proof. This theorem is proved by reviewing earlier statements (for example we notice, that V may be viewed as W_n and that $\dim W_n = \dim W_{-n}$. $\square$

2 Representations of $\mathfrak{sl}_n(\mathbb{C})$

2.1 Constructing a basis for $\mathfrak{sl}_n$

The main difference of studying the $\mathfrak{sl}_n$-case is that we can no longer discuss all $n \times n$ matrices with trace zero explicitly (as we did in the $\mathfrak{sl}_2$-case) but rather we need to classify them with regards to the position of their entries and then analyse the three subalgebras of $\mathfrak{sl}_n$ generated by them:

$$\begin{aligned}
\mathfrak{h} &= \text{Lie algebra of diagonal matrices } H = \operatorname{diag}(\lambda_1, \dots, \lambda_n) \text{ with } \Sigma\ \lambda_i = 0, \lambda_i \in \mathbb{C}, \\
\mathfrak{x} &= \text{Lie algebra of superdiagonal matrices}, \\
\mathfrak{y} &= \text{Lie algebra of infradiagonal matrices}.
\end{aligned}$$

$\mathfrak{sl}_n$ may then be decomposed into their direct sum[2]: $\mathfrak{sl}_n = \mathfrak{h} \oplus \mathfrak{x} \oplus \mathfrak{y}$.
Note that $\mathfrak{h}$ is the cartan subalgebra of $\mathfrak{sl}_n$ (hence abelian[3]), $\mathfrak{x}$ resp. $\mathfrak{y}$ are nilpotent and $\mathfrak{h} \oplus \mathfrak{x}$ is the canonical borel algebra[4].

Definition 2.1.1
Let $\mathfrak{h}^*$ be the dual[5] of $\mathfrak{h}$, then elements $\chi \in \mathfrak{h}^*$ are of the form: $\chi = \sum_{i=1}^{n} u_i \lambda_i$, with $u_i \in \mathbb{C}$ and λ_i being the entries of a diagonal matrix $H \in \mathfrak{h}$.

(i) A linear form $\alpha = \lambda_i - \lambda_j$ $(i < j)$ is called *root*.

(ii) The set of *positive roots* is denoted by $R_+ = \{\alpha \in \mathfrak{h}^* \mid \alpha = \lambda_i - \lambda_j\ ,\ (i < j)\}$,

(iii) The set of by *roots* $R = R_+ \cup (-R_+)$.

(iv) Positive roots of the form: $\alpha_i = \lambda_i - \lambda_{i+1}$ are called *simple roots*.

Using the previous definitions we will now construct more explicit classes of matrices, which will then prove useful to find bases of the subalgebras of $\mathfrak{sl}_n$, making it possible to study them in a way, similar to the case of $\mathfrak{sl}_2$.

Definition 2.1.2
Let $\alpha = \lambda_i - \lambda_j \in R, (i \neq j)$ and $H_\alpha, X_\alpha \in \mathfrak{sl}_n$. We define:

$X_\alpha := X_{(i,j)} = 1$ and zero elsewhere,

$H_\alpha :=\ H \in \mathfrak{h}$ with entries $H_{(i,i)} = 1, H_{(j,j)} = -1$ and zero elsewhere.

[2]Using the concept of *rootspace decomposition* (talk 5) it's possible to construct the following decomposition "from scratch" so to say, but for my talk I will assume it as given
[3]Cf. [Ser87] Ch.3.5 Th.3
[4]The maximal solvable subalgebra
[5]The space of linear forms $\chi : \mathfrak{h} \to \mathbb{C}$

Proposition 2.1.3.

(i) The X_α's make a basis of $\mathfrak{x}$ and the $X_{-\alpha}$'s make a basis of $\mathfrak{y}$.

(ii) If $H \in \mathfrak{h}, \alpha \in R$ then $[H, X_\alpha] = \alpha(H)X_\alpha$.

(iii) $[X_\alpha, X_{-\alpha}] = H_\alpha$.

Proof.

(i) The way the X_α and $X_{-\alpha}$ are defined, the claim follows directly from the conditions $i < j$ and $i \neq j$.

(ii) Consider $(\lambda_1, \dots, \lambda_n)$, the entries on the diagonal of H and $\alpha = \lambda_i - \lambda_j$. Due to the definition of H and X_α we know: $H \cdot X_\alpha = \lambda_i \cdot X_\alpha$ and $X_\alpha \cdot H = \lambda_j \cdot X_\alpha \implies [H, X_\alpha] = (\lambda_i - \lambda_j)X_\alpha = \alpha(H)X_\alpha$.

(iii) $X_\alpha \cdot X_{-\alpha}$ yields a matrix A with $(A_{i,i}) = 1$ and $X_{-\alpha} \cdot X_\alpha = A'$ with $A'_{(j,j)} = 1$. Then we get: $A - A' = H_\alpha$.

□

Remark
The statement $\alpha(H_\alpha) = 2$ is always true.

Example 2.1.4
Returning to the the case of $\mathfrak{sl}_2$ we only have one positive root: $\alpha = \lambda_1 - \lambda_2 = 2$, so H is unique and we can construct the canonical basis:

$$H_\alpha = H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X_\alpha = X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad X_{-\alpha} = Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

2.2 Weights and Primitive Elements of $\mathfrak{sl}_n$

Similar to the way we studied $\mathfrak{sl}_2$ in the first section we will now be analysing the weights and primitive elements of $\mathfrak{sl}_n$-modules to gain insight of their structure. But we will need to update a few definitions in advance:

Definition 2.2.1
Let V be a finite dimensional $\mathfrak{sl}_n$ module, $v \in V,\ \chi \in \mathfrak{h}^*,$ and $H \in \mathfrak{h}$.

(i) For any χ we denote the corresponding space of simultaneous eigenvectors (i.e. $H \cdot v = \chi(H) \cdot v \ \ \forall H \in \mathfrak{h}$) as V_χ and call it *weight space*.

(ii) The elements of the weightspace V_χ are said to have *weight* χ.

(iii) Elements χ with non-empty weight-space V_χ are called *weights* of V.

(iv) The dimension of V_χ is called *multiplicity of* χ.

Proposition 2.2.2. *Let $\chi \in \mathfrak{h}^*,\ v \in V_\chi,\ \alpha \in R$, then $X_\alpha v$ has weight $\chi + \alpha$.*

Proof. This is a simple calculation:
$HX_\alpha v = [H, X_\alpha]v + X_\alpha H v = \alpha(H)X_\alpha v + \chi(H)X_\alpha v = (\alpha + \chi)(H)X_\alpha v \implies X_\alpha v \in V_{\chi+\alpha}$ □

Proposition 2.2.3. *The module V is a direct sum of weightspaces V_χ:*

$$V = \bigoplus_{\chi \in \mathfrak{h}^*} V_\chi$$

Proof. The eigenvectors corresponding to distinct eigenvalues are linearly independent, hence the sum of all weightspaces is direct. We also know that the module V' generated by the sum is stable under $\mathfrak{sl}_n$, due to it being stable by the X_α's and $\mathfrak{h}$. This yields: $V' \subseteq V$. Assume now, that there exists a different, non-zero V'', such that $V' \oplus V'' = V$:
With $\mathfrak{h}$ being abelian[6] and $\mathbb{C}$ being algebraically closed we know that V'' contains an eigenvector $v \neq (0)$ of $\mathfrak{h}$, which by definition should be contained in some V_χ, contradicting the assumption: $V'' \cap V' = 0$. This implies $V' = V$. □

[6]meaning the elements are simultaneous diagonalizable

Definition 2.2.4
$e \in V\backslash\{0\}$ is called *primitive element* if and only if e is an eigenvector to $\mathfrak{h}$ and $X_\alpha e = 0\ \forall \alpha \in R_+$.

Proposition 2.2.5. *Any non-zero $\mathfrak{sl}_n$-module contains a primitive element.*

Proof. Cf. Prop1.1.2. □

We can once again observe, that by applying X_α to elements of a certain weightspace we can again raise or lower their weight similar to the way we let X, Y act in the first section.

2.3 Irreducible $\mathfrak{sl}_n$-Modules

To gain better insight on how to distinctly classify representations of $\mathfrak{sl}_n$, we are going to study modules, that are generated by primitive elements. These are again very similar to the ones mentioned in the first section, but for this too, we will need another, more general concept which will be introduced in the following. As mentioned in the introduction, this is an almost philosophical explanation of the idea, rather than a rigorous definition[7].

2.3.1 The Universal Enveloping Algebra

Definition 2.3.1
A universal enveloping algebra $(U\mathfrak{g}, \pi)$ of $\mathfrak{g}$ is a pair of an associative algebra with unit, and a linear map π, satisfying the following properties:

1. $\pi([X,Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X)\ \forall X, Y \in \mathfrak{g}$
2. $U\mathfrak{g}$ is π-invariant and especially generated by elements $\pi(X)$ $(X \in \mathfrak{g})$, in the sense that there is no Lie algebra, properly contained in $U\mathfrak{g}$ which also contains every $\pi(X)$.
3. For every other associative Lie algebra $\mathfrak{a}$ with unit and a linear map ρ, which satisfies the given "commutator condition", there exists a homomorphism $\phi : U\mathfrak{g} \to \mathfrak{a}$ such that $\phi(1) = 1$ and $\phi(\pi(X) = \rho(X)$.

Fact 2.3.2
The representations of $U\mathfrak{g}$ correspond to those of $\mathfrak{g}$.

Example 2.3.3
Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ with the basis as defined above. The universal algebra $U\mathfrak{g}$ is then given by the associative algebra with unit, generated by three elements x, y, h, satisfying *only* the relations:

$$\begin{aligned} hx - xh &= 2x, \\ hy - yh &= -2y, \\ xy - yx &= h, \end{aligned}$$

and a linear map $\pi : \mathfrak{sl}_n \to U\mathfrak{g}$, such that $\pi(x) = X,\ \pi(y) = Y,\ \pi(H) = h$.

2.4 $\mathfrak{sl}_n$-Modules Generated by Primitive Elements

Let V be an arbitrary $\mathfrak{sl}_n$-module, $e \in V_\chi$ a primitive element and $V_1 = (U\mathfrak{sl}_n) \cdot e$ the module generated by e.

Fact 2.4.1
The weights of V_1 are of the form $\chi - \sum_{i=1}^{n-1} m_i\alpha_i$ $(m_i \geq 0)$.

Proof. We will not prove this rigorously but it follows from the fact, that $U\mathfrak{sl}_n$ can be decomposed into the tensor product of the universal algebra of $\mathfrak{y}$ and the borel-algebra $\mathfrak{b}$, and that the universal algebra of $\mathfrak{y}$ is generated by *monomials*. Cf. [Ser92] Ch.7, Th. 3.1. □

[7] Cf. [Hal15] Sect. 9.3

Theorem 2.4.2.

(i) Any primitive element $v \in V_1$ of weight χ is a multiple of e.

(ii) V_1 is irreducible.

Proof.

(i) Follows from the proof by construction of 2.4.1.

(ii) Suppose: $V_1 = V' \oplus V''$, and $v = v' + v''$.
Consider the weightspace $(V_1)_\chi = V'_\chi \oplus V''_\chi$ which implies that v' and v'' are both of weight χ. As we know from (i) they must be multiples of v, hence, one must be zero (we will choose $v'' = 0$). We then get: $v' = v \implies V' = V_1 \implies V'' = 0$.

□

Theorem 2.4.3. *Let V, V', V'' be irreducible $\mathfrak{sl}_n$-modules.*

(i) V contains a unique primitive element (up to multiplication by elements of $\mathbb{C}$). The weight of this element is specified to be the highest weight of V.

(ii) If V', V'' have the same highest weight, they are isomorphic.

Proof.

(i) V contains at least one primitive element (2.2.5). Let v, v' be primitive elements with weight χ and χ' respectively.
From 2.4.2 we have:

$$\chi - \chi' = \sum_{i=1}^{n-1} m_i \alpha_i \ , \tag{1}$$

$$\chi' - \chi = \sum_{i=1}^{n-1} m'_i \alpha_i \qquad (m_i, m'_i \geq 0 \forall i) \tag{2}$$

which implies $m_i = m'_i = 0 \implies \chi = \chi'$. The scalar multiplicity follows directly from 2.4.2.(ii).

(ii) Let $v' \in V'$ and $v'' \in V''$ be the respective primitive elements, each of weight χ. Consider $V' \oplus V''$ and the corresponding primitive element $v = (v', v'')$, which is also of weight χ. The $\mathfrak{sl}_n$-submodule W of $V' \times V''$ generated by v is irreducible (2.4.2) and the projection map $\pi_i : W \to V_i$ is non-zero. According to Schur's Lemma[8] such an homomorphism between irreducible Lie algebra modules is either an isomorphism or zero, hence V', V'' are both isomorphic to W and therefore $V' \cong V''$.

□

2.5 Classification of Irreducible $\mathfrak{sl}_n$-Modules

After stating Theorem 2.4.3 the only thing we need in order to classify all irreducible $\mathfrak{sl}_n$ modules uniquely, is a way of determining the highest weight of an arbitrary, irreducible $\mathfrak{sl}_n$-module.
Let $\chi \in \mathfrak{h}$, then $\chi(\lambda_1, \ldots, \lambda_n) = u_1\lambda_1 + \ldots + u_n\lambda_n$.

Theorem 2.5.1. *An irreducible $\mathfrak{sl}_n$−module with highest weight χ exists if and only if the difference of coefficients u_i and u_j is a positive integer for all $i < j$.*

To prove this theorem there is a bit of groundwork to do first:

[8]Cf. [Hal15]Ch. 4.5 Th.4.29

Proof of necessity

Let V be an irreducible $\mathfrak{sl}$-module with primitive element e of weight χ. We know, that there is an H_α such that $u_i - u_j = \chi(H_\alpha)$ for the positive root $\alpha = \lambda_i - \lambda_j \in R_+$. Using this it suffices to prove that $\chi(H_\alpha)$ is an integer under the given conditions. First off we will prove, that since V is an $\mathfrak{sl}_n$ module the same (or at least similar) formulas will hold, as we did when studying $\mathfrak{sl}_2$:

Lemma 2.5.2. *Let V be an irreducible $\mathfrak{sl}_n$-module, $e_0 \in V_\chi$ a primitive element and $e^\alpha_m = (\frac{1}{m!})X^m_{-\alpha} \cdot e_0$ then:*

(i) $H \cdot e^\alpha_m = (\chi - m\alpha)(H)e^\alpha_m$,

(ii) $X_{-\alpha} \cdot e^\alpha_m = (m+1)e^\alpha_{m+1}$,

(iii) $X_\alpha \cdot e^\alpha_m = (\chi(H_\alpha) - m + 1)e^\alpha_{m-1}$.

Proof.

(i) This formula basically states the fact, that $e^\alpha_m \in V_{\chi - m\alpha}$, which directly follows from the way $X_{-\alpha}$[9] acts on elements in V_χ.

(ii) $X_{-\alpha}e^\alpha_m = X_{-\alpha} \cdot \frac{1}{m!}X^m_{-\alpha} \cdot e_0 = (m+1)\frac{1}{(m+1)!}X^{m+1}_{-\alpha}e_0 = (m+1)e^\alpha_{m+1}$.

(iii) This is proved via induction on m:
For m=0 the formula holds[10].

$$\begin{aligned} m \cdot X_\alpha e^\alpha_m &= X_\alpha X_{-\alpha} e^\alpha_{m-1} = [X,Y]e^\alpha_{m-1} + X_{-\alpha}X_\alpha e^\alpha_{m-1} \\ &= (\chi(H_\alpha) - (m-1)\alpha(H_\alpha))e^\alpha_{m-1} + (m-1)(\chi(H_\alpha) - m + 2)e^\alpha_{m-1} \\ &= m(\chi(H_\alpha) - m + 1)e^\alpha_{m-1} \end{aligned}$$

The last step uses the earlier remark: "$\alpha(H_\alpha) = 2$ is always true".

□

Observation
As any module V is assumed to be finite dimensional, the number of possible weights is finite as well, hence there must be an integer m, such that $e^\alpha_{m+1} = 0$.
If we combine this observation with formula (iii) we get:

$$X_\alpha e^\alpha_{m+1} = 0 = (\chi(H_\alpha - m)e^\alpha_m \implies \chi(H_\alpha) = m.$$

Proof of sufficiency

Now we need to prove, that there is a $\mathfrak{sl}_n$-module of highest weight χ, under the assumption that the pairwise difference of all coefficients of χ is an positive integer.
We rewrite the definition of χ by introducing linear forms $\pi_1, \ldots, \pi_{n-1}$ with $\pi_i = \sum_{k=1}^i \lambda_k$ and integers $m_1, \ldots, m_{n-1}$:

$$\chi = \sum_{i=1}^{n-1} m_i \pi_i$$

Proposition 2.5.3. *Let χ, χ' be the highest weights of modules V and V', then:*

(i) $\chi + \chi'$ *is the highest weight of an irreducible module* $W \subseteq V \otimes V'$.

(ii) The set of highest weights is closed under addition.

Proof. Let v, v' be the primitive elements of V, V' and corresponding weight χ, χ'.

(i) If v and v' are the primitive elements of V and V', then $v \otimes v'$ is a primitive element of $V \otimes V'$ of weight $\chi + \chi'$. Due to (Th.2.4.2) the submodule generated by $v \otimes v'$ is an irreducible $\mathfrak{sl}$-module with highest weight $\chi + \chi'$.

(ii) this follows from (i).

□

[9] Taking up the role Y had in the $\mathfrak{sl}_2$-case
[10] by convention $e_{-1} = 0$

We can apply this last proposition to the π_i's we defined earlier and prove that they are the highest weights of their respective submodules.

Proposition 2.5.4. *Let V be $\mathbb{C}^n$, and consider it a $\mathfrak{sl}_n$-module, with V_i being the i-th exterior Power of V $(1 \leq i \leq n-1)$. Then V_i is an irreducible $\mathfrak{sl}_n$-module of highest weight π_i.*

Proof. The canonical basis of V is given by $e_1, \ldots, e_n$ and we define $v_i = e_1 \wedge \ldots \wedge e_i$, which is a primitive element of V_i, with weight π_i. For proving the irreducibility of V_i we apply monomials of the $X_{-\alpha}$'s to v_i, allowing us to obtain any term of the form $e_{m_1} \wedge \ldots \wedge e_{m_i}$ which also concludes the proof of Theorem 2.5.1. $\square$

References

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