Cartan subalgebras

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December 15, 2023

In the following, the ground field is $\mathbb{C}$ and the Lie algebras considered are finite-dimensional.

1 Cartan Subalgebra

1.1 Definition of Cartan subalgebras

Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{a}$ a subalgebra of $\mathfrak{g}$.

Definition 1. *The **normalizer** of $\mathfrak{a}$ in $\mathfrak{g}$ is defined to be the set $\mathfrak{n}(\mathfrak{a})$ of all $x \in \mathfrak{g}$ such that $ad(x)(\mathfrak{a}) \subset \mathfrak{a}$. It is the largest subalgebra of $\mathfrak{g}$ which contains $\mathfrak{a}$ and in which $\mathfrak{a}$ is an ideal.*

Definition 2. *A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called a **Cartan subalgebra** (CSA) of $\mathfrak{g}$ if it satisfies the following two conditions:*

(a) $\mathfrak{h}$ *is nilpotent.*

(b) $\mathfrak{h} = \mathfrak{n}(\mathfrak{h})$

1.2 Examples of Cartan subalgebras

1. Any nilpotent Lie algebra is its own Cartan subalgebra.
2. The algebra $\mathfrak{D}$ of all diagonal matrices is a Cartan subalgebra of $\mathfrak{gl}_\mathfrak{n}$:

   Since $\mathfrak{D}$ is abelian, it is clearly a nilpotent Lie algebra. So it remains to show that $\mathfrak{D} = \mathfrak{n}(\mathfrak{D})$. (For sake of clarity, we consider only the case n=2, but for other cases, the same argument holds.) Let $B \in n(\mathfrak{D})$. For any $A \in \mathfrak{D}$ we get:
   $ad(B)(A) = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \cdot \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} - \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} \cdot \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} = \begin{pmatrix} a_1 \cdot b_{1,1} & a_1 \cdot b_{1,2} \\ a_2 \cdot b_{2,1} & a_2 \cdot b_{2,2} \end{pmatrix} - \begin{pmatrix} a_1 \cdot b_{1,1} & a_2 \cdot b_{1,2} \\ a_1 \cdot b_{2,1} & a_2 \cdot b_{2,2} \end{pmatrix} = \begin{pmatrix} 0 & (a_1 - a_2) \cdot b_{1,2} \\ (a_1 - a_2) \cdot b_{2,1} & 0 \end{pmatrix} \in \mathfrak{D}$
   As this must hold for any $A \in \mathfrak{D}$, it in particular holds when $a_1 \neq a_2$. Thus $b_{1,2} = b_{2,1} = 0$ and $B \in \mathfrak{D}$. Therefore $\mathfrak{n}(\mathfrak{D}) \subset \mathfrak{D}$. Together with $\mathfrak{D} \subset \mathfrak{n}(\mathfrak{D})$ (what clearly holds) $\mathfrak{D} = \mathfrak{n}(\mathfrak{D})$ follows.

3. $\mathfrak{sl}_\mathfrak{n}$ has the Cartan subalgebra $\mathfrak{h}$ of diagonal matrices with trace 0.
   For $n = 2$: $\mathfrak{h} = \{\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} | a \in \mathbb{C}\}$

To show this, one can use similar arguments as above.

4. The Cartan subalgebra of $\mathcal{SO}_{2n}$ is the set S of matrices of the form $\begin{pmatrix} A_1 & ... & 0 \\ & \cdots & \\ 0 & ... & A_n \end{pmatrix}$ (with $A_i = \begin{pmatrix} 0 & a_i \\ -a_i & 0 \end{pmatrix}, a_i \in \mathbb{C}$) as Cartan subalgebra.

   Again, we will show this for the case n=2 and all the other cases follow similarly. We first verify that S is nilpotent. Let $A, B \in S$. $[A,B] =$
   $\begin{pmatrix} 0 & a_1 & 0 & 0 \\ -a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_2 \\ 0 & 0 & -a_2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & b_1 & 0 & 0 \\ -b_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2 \\ 0 & 0 & -b_2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & b_1 & 0 & 0 \\ -b_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2 \\ 0 & 0 & -b_2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & a_1 & 0 & 0 \\ -a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_2 \\ 0 & 0 & -a_2 & 0 \end{pmatrix} =$
   $\begin{pmatrix} a_1 \cdot b_1 - b_1 \cdot a_1 & 0 & 0 & 0 \\ 0 & a_2 \cdot b_2 - b_2 \cdot a_2 & 0 & 0 \\ 0 & 0 & a_3 \cdot b_3 - b_3 \cdot a_3 & 0 \\ 0 & 0 & 0 & a_4 \cdot b_4 - b_4 \cdot a_4 \end{pmatrix} = 0$
   $\implies [\mathfrak{g}, \mathfrak{g}] = 0.$

   Next we want to argue why $\mathfrak{n}(S) = S$. Let $C \in \mathfrak{n}(S)$. For all $D \in S$:
   $ad(C)(D) = \begin{pmatrix} 0 & c_{1,2} & c_{1,3} & c_{1,4} \\ -c_{1,2} & 0 & c_{2,3} & c_{2,4} \\ -c_{1,3} & -c_{2,3} & 0 & c_{3,4} \\ -c_{1,4} & -c_{2,4} & -c_{3,4} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & d_1 & 0 & 0 \\ -d_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_2 \\ 0 & 0 & -d_2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & d_1 & 0 & 0 \\ -d_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_2 \\ 0 & 0 & -d_2 & 0 \end{pmatrix} \cdot$
   $\begin{pmatrix} 0 & c_{1,2} & c_{1,3} & c_{1,4} \\ -c_{1,2} & 0 & c_{2,3} & c_{2,4} \\ -c_{1,3} & -c_{2,3} & 0 & c_{3,4} \\ -c_{1,4} & -c_{2,4} & -c_{3,4} & 0 \end{pmatrix} = \begin{pmatrix} -d_1 \cdot c_{1,2} & 0 & -d_2 \cdot c_{1,4} & d_2 \cdot c_{1,3} \\ 0 & d_1 \cdot -c_{1,2} & -d_2 \cdot c_{2,4} & d_2 \cdot c_{2,3} \\ -d_1 \cdot -c_{2,3} & d_1 \cdot -c_{1,3} & -d_2 \cdot c_{3,4} & 0 \\ -d_1 \cdot -c_{2,4} & d_1 \cdot -c_{1,4} & 0 & d_2 \cdot -c_{3,4} \end{pmatrix} -$
   $\begin{pmatrix} d_1 \cdot -c_{1,2} & 0 & d_1 \cdot c_{2,3} & d_1 \cdot c_{2,4} \\ 0 & -d_1 \cdot c_{1,2} & -d_1 \cdot c_{1,3} & -d_1 \cdot c_{1,4} \\ d_2 \cdot -c_{1,4} & d_2 \cdot -c_{2,4} & d_2 \cdot -c_{3,4} & 0 \\ -d_2 \cdot -c_{1,3} & -d_2 \cdot -c_{2,3} & 0 & -d_2 \cdot c_{3,4} \end{pmatrix} \in S$
   $\implies -d_2 \cdot c_{1,4} - d_1 \cdot c_{2,3} = -(-d_1 \cdot -c_{2,3} - d_2 \cdot -c_{1,4})$, so $2 \cdot d_2 \cdot c_{1,4} = 0$ and therefore (since this holds for any d_2) $c_{1,4} = 0$. Analogously $c_{1,3} = 0$.
   $\implies 0 = -d_2 c_{2,4} + d_1 c_{1,3} = -d_2 c_{2,4}$, so $c_{2,4} = 0$. Analogously $c_{2,3} = 0$.
   $\implies C \in S$ and therefor $\mathfrak{n}(S) = S$.

We will see later that indeed, every Lie algebra has a Cartan subalgebra.

2 Regular Elements: Rank

Let $\mathfrak{g}$ be a Lie algebra.

2.1 The Characteristic Polynomial of ad x

If $x \in \mathfrak{g}$, we will let $P_x(T)$ denote the characteristic polynomial of the endomorphism $ad\,x$ defined by x. We have

$$P_x(T) = det(T - ad(x)). \tag{1}$$

Let $n = dim\ \mathfrak{g}$. We can write $P_x(\mathrm{T})$ in the form

$$P_x(T) = \sum_{i=0}^{n} a_i(x) T^i. \tag{2}$$

If x has coordinates $x_1, ..., x_n$ (with respect to a fixed basis of g), we can view $a_i(x)$ as a function of the n complex variables $x_1, ..., x_n$. It is a homogeneous polynomial of degree $n-1$ in $x_1, ..., x_n$.

2.2 The Rank and Regular Elements

Definition 3. *The **rank** of $\mathfrak{g}$ is the least integer l such that the function a_l is not idendically zero.*

Since $a_n = 1$, we must have $l \le n$ with equality iff g is nilpotent.
On the other hand, if x is a nonzero element of $\mathfrak{g}$ then $ad(x)(x) = 0$, showing that 0 is an eigenvalue of $ad\ x$. It follows that if $\mathfrak{g} \neq 0$ then $a_0 = 0$, so that $l \geq 1$.

Definition 4. *An element $x \in \mathfrak{g}$ is said to be **regular** if $a_l(x) \neq 0$.*

3 The Cartan Subalgebra Associated with a Regular Element

Let $\mathfrak{g}$ be a Lie algebra.

Definition 5. *Let x be an element of $\mathfrak{g}$. If $\lambda \in \mathbb{C}$, we let $\mathfrak{g}_x^\lambda$ denote the set of $y \in \mathfrak{g}$ such that $(ad(x) - \lambda)^p y = 0$ for sufficiently large p and call it the **nilspace** of $ad(x) - \lambda$.*

In particular, $\mathfrak{g}_x^0$ is the nilspace of $ad\ x$. Its dimension is the multiplicity of 0 as an eigenvalue of $ad\ x$; that is, the least integer i such that $a_i(x) \neq 0$.

Proposition 1. *Let $x \in \mathfrak{g}$. Then:*

(a) $\mathfrak{g}$ *is the direct sum of the nilspaces* $\mathfrak{g}_x^\lambda$

(b) $[\mathfrak{g}_x^\lambda, \mathfrak{g}_x^\mu] \subset \mathfrak{g}_x^{\lambda+\mu}$ *if* $\lambda, \mu \in \mathbb{C}$.

(c) $\mathfrak{g}_x^0$ *is a Lie subalgebra of* $\mathfrak{g}$.

Remark: The inclusion in (b) is in general no equality. For instance, for $\mu = \lambda$ the left-hand side is zero, while the right-hand side might be larger.
For a concrete example, consider $x := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{gl}_2(\mathbb{C}) =: \mathfrak{g}, \mu = \lambda = 0$:
$ad(x)(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} 0 & 2b \\ -2c & 0 \end{pmatrix} \implies \mathfrak{D} \subset \mathfrak{g}_x^0$ (here b=c=0). For general p $(ad(x)^p(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} 0 & 2^p b \\ (-2)^p c & 0 \end{pmatrix}$. For the right-hand side to become 0, b and c need to be 0. $\implies \mathfrak{g}_x^0 \subset \mathfrak{D}$.
Then $\mathfrak{g}_x^{\lambda+\mu} = \mathfrak{g}_x^\lambda = \mathfrak{g}_x^\mu = \mathfrak{g}_x^0 = \mathfrak{D}$ is the nilspace of $ad(x)$. But since diagonal matrices commute $[\mathfrak{g}_x^\lambda, \mathfrak{g}_x^\mu] = [\mathfrak{g}_x^0, \mathfrak{g}_x^0] = \{0\}$.

Proof.

(a) This is a standard property of vectorspace endomorphisms applied to adx.

(b) Let $y \in \mathfrak{g}_x^\lambda, z \in \mathfrak{g}_x^\mu$. We want to show that, then $[y,z] \in \mathfrak{g}_x^{\lambda+\mu}$. Now we can use induction to prove the following formula:

$$(ad\,x - \lambda - \mu)^n[y,z] = \sum_{p=0}^{n} \binom{n}{p} [(ad\,x - \lambda)^p y, (ad\,x - \mu)^{n-p} z] \tag{3}$$

BC: $(ad\,x - \lambda - \mu)^0[y,z] = [y,z] = \sum_{p=0}^{0} \binom{0}{p} [(ad\,x - \lambda)^p y, (ad\,x - \mu)^{0-p} z]$.

IS: For simplicity $c_n(p) := [(ad\,x - \lambda)^p y, (ad\,x - \mu)^{n-p} z]$.

$\sum_{p=0}^{n+1} \binom{n+1}{p} [(ad\,x - \lambda)^p y, (ad\,x - \mu)^{n+1-p} z] =$
$\sum_{p=1}^{n} \binom{n+1}{p} c_{n+1}(p) + c_{n+1}(0) + c_{n+1}(n+1) =$
$\sum_{p=1}^{n} (\binom{n}{p} c_{n+1}(p) + \binom{n}{p-1} c_{n+1}(p)) + c_{n+1}(0) + c_{n+1}(n+1) =$
$\sum_{p=1}^{n} (\binom{n}{p} c_{n+1}(p)) + c_{n+1}(0) + \sum_{p=0}^{n-1} (\binom{n}{p} c_{n+1}(p+1)) + c_{n+1}(n+1) =$
$\sum_{p=0}^{n} \binom{n}{p} (c_{n+1}(p) + c_{n+1}(p+1)) =$
$\sum_{p=0}^{n} \binom{n}{p} ([(ad\,x - \lambda)^p y, (ad\,x - \mu)^{n+1-p} z] + [(ad\,x - \lambda)^{p+1} y, (ad\,x - \mu)^{n-p} z])=$
$\sum_{p=0}^{n} \binom{n}{p} ([(ad\,x - \lambda)^p y, [x, (ad\,x - \mu)^{n-p} z]] - \mu[(ad\,x - \lambda)^p y, (ad\,x - \mu)^{n-p} z] + [[x, (ad\,x - \lambda)^p y], (ad\,x - \mu)^{n-p} z] - \lambda[(ad\,x - \lambda)^p y, (ad\,x - \mu)^{n-p} z])=$
(By Jacobi-Identity)
$\sum_{p=0}^{n} \binom{n}{p} ([x, [(adx-\lambda)^p y, (adx-\mu)^{n-p} z]] - (\mu+\lambda)[(adx-\lambda)^p y, (adx-\mu)^{n-p} z])=$
$(ad\,x - \lambda - \mu)(\sum_{p=0}^{n} \binom{n}{p} [(ad\,x - \lambda)^p y, (ad\,x - \mu)^{n-p} z])$
(By IH)
$= (ad\,x - \lambda - \mu)^{n+1}[y,z]$.

If we now take n sufficiently large in the just proven formula, all terms on the right vanish, showing that [y, z] is indeed in $\mathfrak{g}_x^{\lambda+\mu}$.

(c) Follows from (b), applied to the case $\lambda = \mu = 0$.

□

Theorem 1. *If x is regular, $\mathfrak{g}_x^0$ is a Cartan subalgebra of $\mathfrak{g}$; its dimension is equal to the rank l of $\mathfrak{g}$.*

This provides a construction for Cartan subalgebras; we shall see that, in fact, it gives all of them.

4 Conjugacy of Cartan Subalgebras

Let $\mathfrak{g}$ be a Lie algebra. We let G denote the inner automorphism group of $\mathfrak{g}$ that is, the subgroup of $Aut(\mathfrak{g})$ generated by the automorphisms $e^{ad(y)}$ for $y \in \mathfrak{g}$.

Theorem 2. *The group G acts transitively on the set of CSAs of $\mathfrak{g}$.*

Combining both theorems, we deduce:

Corollary 1. *The dimension of a CSA of* $\mathfrak{g}$ *is equal to the rank of* $\mathfrak{g}$.

Corollary 2. *Every CSA of* $\mathfrak{g}$ *has the form* $\mathfrak{g}_x^0$ *for some regular element* x *of* $\mathfrak{g}$.

5 The Semisimple Case

Theorem 3. *Let* $\mathfrak{h}$ *be a CSA of a semisimple Lie algebra* $\mathfrak{g}$. *Then:*

(a) The restriction of the Killing form of $\mathfrak{g}$ *to* $\mathfrak{h}$ *is nondegenerate.*

(b) $\mathfrak{h}$ *is abelian.*

(c) The centralizer of $\mathfrak{h}$ *is* $\mathfrak{h}$.

(d) Every element of $\mathfrak{h}$ *is semisimple.*

Proof.

(a) By Corollary 2 to Theorem 2, there is a regular element x such that $\mathfrak{h} = \mathfrak{g}_x^0$. Let $\mathfrak{g} = \mathfrak{g}_x^0 \oplus \sum_{\lambda \neq 0} \mathfrak{g}_x^\lambda$ be the canonical decomposition of $\mathfrak{g}$ with respect to x (cf. Prop. 1).
Let B denote the Killing form of $\mathfrak{g}$. Then by applying Cartan's criterion to $\mathfrak{h}$ and to the representation ad:$\mathfrak{h} \to End(g)$, we see that $Tr(adx \circ ady) = 0$ for $x \in \mathfrak{h}$ and $y \in [\mathfrak{h}, \mathfrak{h}]$. So together with proposition 1 for $\lambda, \mu \in \mathbb{C}$ with $\lambda + \mu \neq 0$: $B(\mathfrak{g}_x^\lambda, \mathfrak{g}_x^\mu) = B([\mathfrak{h}, \mathfrak{g}_x^\lambda], \mathfrak{g}_x^\mu) = B(\mathfrak{h}, [\mathfrak{g}_x^\lambda, \mathfrak{g}_x^\mu]) \subset B(\mathfrak{h}, \mathfrak{g}_x^{\lambda+\mu}) = B(\mathfrak{h}, [\mathfrak{h}, \mathfrak{g}_x^{\lambda+\mu}]) = B([\mathfrak{h}, \mathfrak{h}], \mathfrak{g}_x^{\lambda+\mu}) = 0$. This shows that $\mathfrak{g}_x^\lambda$ and $\mathfrak{g}_x^\mu$ are orthogonal with respect to B.
We therefore have a decomposition of $\mathfrak{g}_x$ into mutually orthogonal subspaces $\mathfrak{g} = \mathfrak{g}_x^0 \oplus \sum_{\lambda \neq 0} (\mathfrak{g}_x^\lambda \oplus \mathfrak{g}_x^{-\lambda})$. Since B is nondegenerate, so is its restriction to each of these subspaces, giving (a) since $\mathfrak{h} = \mathfrak{g}_x^0$.

(b) Like above $Tr(adx \circ ady) = 0$ for $x \in \mathfrak{h}$ and $y \in [\mathfrak{h}, \mathfrak{h}]$. In other words, $[\mathfrak{h}, \mathfrak{h}]$ is orthogonal to $\mathfrak{h}$ with respect to the Killing form B. Because of (a), this implies that $[\mathfrak{h}, \mathfrak{h}] = 0$.

(c) Being abelian, $\mathfrak{h}$ is contained in its own centralizer $\mathfrak{c}(\mathfrak{h})$. Moreover, $\mathfrak{c}(\mathfrak{h})$ is clearly contained in the normalizer $\mathfrak{n}(\mathfrak{h})$ of $\mathfrak{h}$. Since $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$, we have $\mathfrak{c}(\mathfrak{h}) = \mathfrak{h}$.

(d) Let $x \in \mathfrak{h}$ and let s (resp. n) be its semisimple (resp. nilpotent) component. If $y \in \mathfrak{h}$, then y commutes with x. By construction, n and s are polynomials of x and an element that commutes with x also so commutes with any polynomial in x, hence also with s and n. So y commutes with both n and s. We therefore have $s, n \in \mathfrak{c}(\mathfrak{h}) = \mathfrak{h}$. However, since y and n commute and $ad(n)$ is nilpotent, $ad(y) \circ ad(n)$ is also nilpotent and its trace $B(y, n)$ is zero. Thus n is orthogonal to every element of $\mathfrak{h}$. Since it belongs to $\mathfrak{h}$, n is zero by (a). Thus $x = s$ which shows that x is indeed semisimple.

□

From (b) follows:

Corollary 1. $\mathfrak{h}$ *is a maximal abelian subalgebra of* $\mathfrak{g}$*.*

Since any regular element is contained in a Cartan subalgebra of $\mathfrak{g}$, we get:

Corollary 2. *Every regular element of* $\mathfrak{g}$ *is semisimple.*

One can show that every maximal abelian subalgebra of g consisting of semisimple elements is a Cartan subalgebra of $\mathfrak{g}$. However, if $\mathfrak{g} \neq 0$ there are maximal abelian subalgebras of $\mathfrak{g}$ which contain nonzero nilpotent elements, and are therefore not Cartan subalgebras.