University of Heidelberg
Seminar on Lie Algebras

Simple and semisimple Lie Algebras

Second talk

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1.1 Simple Lie Algebras

Definition 1.1.1.
We call a Lie Algebra $\mathfrak{g}$ *simple if:*
(i) $\mathfrak{g}$ *is non-abelian, and*
(ii) the only ideals of $\mathfrak{g}$ *are* $\{0\}$ *or* $\mathfrak{g}$ *itself.*

1.2 Semisimple Lie Algebras

Definition 1.2.1.
We call a Lie Algebra $\mathfrak{g}$ *semisimple if:*
(i) All abelian ideals of $\mathfrak{g}$ *are* $\{0\}$ $\Longleftrightarrow$ *(ii) the radical* $\mathfrak{r}$ *of* $\mathfrak{g}$ *being* $\{0\}$.

Example 1.2.2. *:*
(i) $\mathfrak{sl}_n(V)$ *is semisimple.*
(ii) $\mathfrak{sl}_2(V)$ *is semisimple.*
(iii) $\mathfrak{so}_{2n+1}(V)$ *is semisimple.*
(iv) $\mathfrak{so}_{2n}(V)$ *is semisimple.*
(v) $\mathfrak{sp}_{2n}(V)$ *is semisimple.*
(vi) $\mathfrak{gl}_n(V)$ *is not semisimple.*

We will continue with $\mathfrak{sl}_2(V)$, $\mathfrak{sl}_n(V)$, and $\mathfrak{gl}_n(V)$ in Chapter 1.4.

1.3 Cartan's Criterion

Lemma 1.3.1. *Let* $A, B \subseteq \mathfrak{gl}_n(V)$ *with* $\dim(V) < \infty$. *Define* $M = \{x \in \mathfrak{gl}_n(V) \mid [x, B] \subseteq A\}$. *Now suppose for* $x \in M$: $\mathrm{Tr}(xy) = 0$ *for all* $y \in M \implies x$ *is nilpotent.*

Proof. Proof in Humphreys, Lemma 4.3 Cartan's Criterion (Page 19). □

Theorem 1.3.2 (Cartan's Criterion)**.** *Let* L *be a subalgebra of* $\mathfrak{gl}_n(V)$ *with* $\dim(V) < \infty$. *Suppose* $\mathrm{Tr}(xy) = 0$ *for all* $x \in [L, L]$ *and for all* $y \in L \implies L$ *is solvable.*

Proof. Now we need to prove that $[L, L]$ is nilpotent, which means that all $x \in [L, L]$ are nilpotent elements. We apply Lemma 1.3.1 on V, any vector space, with $A = [L, L]$, $B = L$:

$$M = \{x \in \mathfrak{gl}_n(\mathbb{F}) \mid [x, L] \subseteq [L, L]\} \supseteq L,$$
$$\implies L/[L, L] \text{ is abelian.}$$

We need to show: $x \in [L, L] \implies \mathrm{Tr}(xy) = 0$ for all $y \in L \implies$ x is nilpotent.
Suppose $[x, y]$ is the generator of $[L, L]$, $z \in M$:

$$\implies \mathrm{Tr}([x, y]z) = \mathrm{Tr}(x[y, z]) = \mathrm{Tr}([y, z]x) = 0,$$

the last equation holds because $[y, z] \in [L, L] \implies x$ is nilpotent.

□

Corollary 1.3.3. *Let* L *be a Lie algebra with* $\mathrm{Tr}(\mathrm{ad}(x), \mathrm{ad}(y)) = 0$ *for all* $x \in [L, L]$ *and for all* $y \in L \implies L$ *is solvable.*

Proof. This corollary follows directly by applying Theorem 1.3.2 (Cartan's Criterion) and Proposition 3.1 in Humphreys (Page 11). □

1.4 Killing Form

Definition 1.4.1 (Killing Form)**.**
Let L be a Lie Algebra and $x, y \in L$. We define the Killing form $\mathfrak{K}_L$ as:

$$\mathfrak{K}_L = \mathrm{Tr}(\mathrm{ad}x\, \mathrm{ad}y)$$

Then $\mathfrak{K}_L$ is a bilinear form on our Lie Algebra L, called the Killing form.

Remark 1.4.2.
(i) $\mathfrak{K}_L$ is associative, meaning that $\forall x, y, z \in L$: $\mathfrak{K}_L([x, y], z) = \mathfrak{K}_L(x, [y, z])$.

We can instantly see that from the properties of the trace (i.e., $\mathrm{Tr}([x, y], z) = \mathrm{Tr}(x, [y, z])$).

(ii) $\mathfrak{K}_L$ is skew-symmetric, which holds the following property $\forall x, y \in L$: $\mathfrak{K}_L(x, y) = -\mathfrak{K}_L(y, x)$.

Lemma 1.4.3.
Let $I \subseteq L$ be an ideal of our Lie Algebra, $\mathfrak{K}_L$ the Killing form over L, and $\mathfrak{K}_I$ the Killing form of I. Then:

$$\implies \mathfrak{K}_I = \mathfrak{K}_L|_I$$

Proof.
If $\Phi : V \to W$ is a linear map, then $\mathrm{Tr}_V(\Phi) = \mathrm{Tr}_W(\Phi|_W)$. Let $x \in I$, so $\mathrm{ad}_x(L) \subseteq I$. In matrix notation we get:

$$\mathrm{ad}_x = \begin{pmatrix} 0 & 0 \\ * & \mathrm{ad}_x|_I \end{pmatrix}$$

For $x, y \in I$, we have:

$$\implies \mathrm{ad}_x \circ \mathrm{ad}_y = \begin{pmatrix} 0 & 0 \\ * & \mathrm{ad}_x|_I \end{pmatrix} \begin{pmatrix} 0 & 0 \\ * & \mathrm{ad}_y|_I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ * & \mathrm{ad}_x|_I \circ \mathrm{ad}_y|_I \end{pmatrix}$$

$$\implies \mathfrak{K}_L|_I = \mathrm{Tr}(\mathrm{ad}_x \circ \mathrm{ad}_y|_I) = \mathrm{Tr}(\mathrm{ad}_x|_I \circ \mathrm{ad}_y|_I) = \mathfrak{K}_I$$

□

1.4.1 Criterion for Semisimplicity

Theorem 1.4.1 (Cartan-Killing Criterion)**.** *A Lie Algebra L is called semisimple* $\iff$ $\mathfrak{K}(X,Y)$ *is nondegenerate.*

Proof.
First, we have to prove that if $\text{Rad}(L) = 0 \implies \text{Rad}(\mathfrak{K}_L) = 0$, where $\text{Rad}(\mathfrak{K}_L) = \{x \in L \mid \mathfrak{K}_L(x,y) = 0 \, \forall y \in L\}$. For this, we need to assume that $\text{Rad}(\mathfrak{K}_L)$ is an ideal of L, meaning $\forall x \in L$ and $y \in \text{Rad}(\mathfrak{K}_L) \implies [x,y] \in \text{Rad}(\mathfrak{K}_L)$. Let's take any $z \in L$:

$$\mathfrak{K}_L([x,y],z) = \mathfrak{K}_L(x,[y,z])$$

Since $y \in \text{Rad}(\mathfrak{K}_L)$, keep in mind that $\mathfrak{K}_L(y,[z,x]) = 0$. We continue the computation:

$$\mathfrak{K}_L([x,y],z) = \mathfrak{K}_L(x,[y,z]) = \mathfrak{K}_L([y,z],x) = \mathfrak{K}_L(y,[z,x]) = 0$$

$\implies [x,y] \in \text{Rad}(\mathfrak{K}_L)$, meaning $\text{Rad}(\mathfrak{K}_L)$ is an ideal of L.
We can extend the statement to $\text{ad}_L(\text{Rad}(\mathfrak{K}_L))$ being a solvable ideal in $\text{ad}_L(L)$ using Theorem 1.3.2 (Cartan's Criterion). We get:

$$\begin{aligned}\text{Tr}(\text{ad}x, \text{ad}y) = \mathfrak{K}_L = 0 \, \forall x \in \text{Rad}(\mathfrak{K}_L) \supseteq [\text{Rad}(\mathfrak{K}_L), \text{Rad}(\mathfrak{K}_L)] \\ \forall y \in L \supseteq \text{Rad}(\mathfrak{K}_L)\end{aligned}$$

$\mathfrak{z}(\text{Rad}(\mathfrak{K}_L)) = \{z \in \text{Rad}(\mathfrak{K}_L) \mid \forall x \in \text{Rad}(\mathfrak{K}_L) : [x,z] = 0\}$ is abelian and solvable. Since:

$$\text{ad}_L(\text{Rad}(\mathfrak{K}_L)) \cong \text{Rad}(\mathfrak{K}_L)/\mathfrak{z}(\text{Rad}(\mathfrak{K}_L))$$

we conclude that $\text{Rad}(\mathfrak{K}_L)$ is solvable.
As $\text{Rad}(\mathfrak{K}_L) \subseteq \text{Rad}(L) \implies \text{Rad}(\mathfrak{K}_L) = 0$, this means that $\mathfrak{K}_L$ is nondegenerate.
The second statement we need to show is that if $\text{Rad}(\mathfrak{K}_L) = 0 \implies \text{Rad}(L) = 0$.
Let's assume L is not a semisimple Lie Algebra, so there exists an abelian ideal $I \subseteq L$:

$$\begin{aligned}&\implies (\text{ad}x\,\text{ad}y)^2 = 0, \; x \in L, y \in I \\ &\implies (\text{ad}x\,\text{ad}y) \text{ is nilpotent} \\ &\implies \mathfrak{K}_L = \text{Tr}(\text{ad}x\,\text{ad}y) = 0 \, \forall x \in L, y \in I \\ &\implies 0 \neq I \subseteq \text{Rad}(\mathfrak{K}_L) \\ &\implies \mathfrak{K}_L \text{ is degenerate}\end{aligned}$$

□

1.4.2 Example on $\mathfrak{sl}_n(\mathbb{F})$

Remark 1.4.1 (continuation of Example 1.2.2).
$\mathfrak{sl}_n(\mathbb{F})$ is a semisimple Lie Algebra.

Proof.
In the first part of the proof, we focus on showing that $k_{\mathfrak{sl}_n}(x, y) = 2n\mathrm{Tr}(xy)$ for all $x, y \in \mathfrak{sl}_n(\mathbb{F})$, which will be helpful to demonstrate that the Killing form is nondegenerate: If we examine the Killing form over $\mathfrak{gl}_n(\mathbb{F})$ with the basis $\{E_{ij}\}$, we obtain:

$$\begin{aligned}
\mathrm{ad}E_{ij}\,\mathrm{ad}E_{kl}(E_{gh}) &= [E_{ij}, E_{kl}E_{gh} - E_{gh}E_{kl}] \\
&= [E_{ij}, \delta_{lg}E_{kh} - \delta_{hk}E_{gl}] \\
&= E_{ij}(\delta_{lg}E_{kh} - \delta_{hk}E_{gl}) - (\delta_{lg}E_{kh} - \delta_{hk}E_{gl})E_{ij} \\
&= \delta_{lg}E_{ij}E_{kh} - \delta_{hk}E_{ij}E_{gl} - \delta_{lg}E_{kh}E_{ij} - \delta_{hk}E_{gl}E_{ij} \\
&= \delta_{lg}\delta_{jk}E_{ih} - \delta_{hk}\delta_{jg}E_{il} - \delta_{lg}\delta_{hi}E_{kj} - \delta_{hk}\delta_{li}E_{gj}
\end{aligned}$$

Now, we focus on the (g, h) coordinate of the vector $\mathrm{ad}(E_{ij}\,\mathrm{ad}(E_{kl})(E_{gh}))$ in the basis of $(E_{\alpha\beta})_{1\leq\alpha,\beta\leq n}$:

$$a_{gh} = \delta_{gi}\delta_{lg}\delta_{jk} - \delta_{ig}\delta_{lh}\delta_{hk}\delta_{jg} - \delta_{kg}\delta_{jh}\delta_{lg}\delta_{hi} - \delta_{jh}\delta_{gk}\delta_{li}$$

$$\begin{aligned}
\implies k_{\mathfrak{gl}_n}(E_{ij}, E_{kl}) &= \sum_{g=1}^{n} a_{gg} \\
&= \sum_{g=1}^{n}(\delta_{gi}\delta_{lg}\delta_{jk} - \delta_{ig}\delta_{lg}\delta_{gk}\delta_{jg} - \delta_{kg}\delta_{jg}\delta_{lg}\delta_{gi} - \delta_{jg}\delta_{gk}\delta_{li}) \\
&= n\delta_{il}\delta_{jk} - \delta_{kl}\delta_{ij} - \delta_{ij}\delta_{kl} + n\delta_{jk}\delta_{il} \\
&= 2n\delta_{il}\delta_{jk} + 2\delta_{ij}\delta_{kl} \\
&= 2n\mathrm{Tr}(E_{ij}E_{kl}) - 2\mathrm{Tr}(E_{ij})\mathrm{Tr}(E_{kl})
\end{aligned}$$

Using Lemma 1.4.3, $\mathrm{Tr}(x) = 0$ for all $x \in \mathfrak{sl}_n(\mathbb{F})$, the bilinearity of the Killing form, and the fact that $\mathfrak{sl}_n$ is an ideal of $\mathfrak{gl}_n$, we get:
$\implies k_{\mathfrak{sl}_n} = k_{\mathfrak{gl}_n}|_{\mathfrak{sl}_n} \implies k_{\mathfrak{sl}_n}(x, y) = 2n\mathrm{Tr}(xy) - 2\mathrm{Tr}(x)\mathrm{Tr}(y) = 2n\mathrm{Tr}(xy)$

It remains to show that $k_{\mathfrak{sl}_n}$ is nondegenerate, so we can use Theorem 1.4.1: Let's assume we have a nonzero $x \in \mathfrak{sl}_n(\mathbb{F})$ such that $k_{\mathfrak{sl}_n}(x, y) = 2n\mathrm{Tr}(xy) = 0$ for all $y \in \mathfrak{sl}_n(\mathbb{F})$. Considering $y = x$, since $x \neq 0 \implies x^2 \neq 0 \implies k_{\mathfrak{sl}_n}(x, x) = 2n\mathrm{Tr}(x^2) \neq 0$, which is a contradiction. □

1.4.3 Example on $\mathfrak{gl}_n(\mathbb{F})$

Remark 1.4.1 (continuation of Example 1.2.2)**.**
$\mathfrak{gl}_n(\mathbb{F})$ is not a semisimple Lie Algebra.

Proof. From the previous remark, we know that $k_{\mathfrak{gl}_n}(x, y) = 2n\text{Tr}(xy) - 2\text{Tr}(x)\text{Tr}(y)$. Let's use $x = \lambda I$ for all $y \in \mathfrak{gl}_n$, which implies $\text{Tr}(x) = n\lambda$ and $xy = yx = \lambda y$.

$$\implies k_{\mathfrak{gl}_n}(\lambda I, y) = 2n\text{Tr}(\lambda y) - 2\text{Tr}(\lambda I)\text{Tr}(y) = 2n\lambda\text{Tr}(y) - 2n\lambda\text{Tr}(y) = 0$$

$\implies k_{\mathfrak{gl}_n}$ is degenerate. □

1.5 Simple Ideals of a Semisimple Lie Algebra

Definition 1.5.1 (Direct Sum)**.**
A Lie Algebra L is a direct sum of ideals $I_1, I_2, ..., I_n$ if:

$$L = I_1 \oplus I_2 \oplus \ldots \oplus I_n$$

This means that each $x \in L$ can be uniquely represented as $x = x_1 + x_2 + \ldots + x_n$ with $x_j \in I_j$, and $I_j \cap I_k = \{0\}$ for all $j \neq k$.

Theorem 1.5.2.
If L is a semisimple Lie Algebra, then there exist ideals $L_1, L_2, ..., L_n \subseteq L$ such that:

$$(i)\ \text{Each } L_i \text{ is simple,}$$
$$(ii) L = L_1 \oplus L_2 \oplus \ldots \oplus L_n.$$

Proof. □

Bibliography

[Hum72] James E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, Vol. 9, Springer-Verlag, New York-Berlin, 1972.

[Ser87] Jean-Pierre Serre, *Complex semisimple Lie algebras*, Springer-Verlag, New York, 1987, Translated from the French by G. A. Jones.