Nilpotent and solvable Lie algebras

Seminar on Lie algebras
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first talk: Nilpotent and solvable Lie algebras
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I. Basic notions

Let F be an arbitrary field throughout the discussion.

Definition I.1 *(Lie algebra)*
*A **Lie algebra** L is a vector space L over a field F together with a binary operation*

$$[\cdot,\cdot] : L \times L \to L,$$

*called the corresponding (Lie-)**bracket** satisfying the following axioms:*

$(L1)$ *The bracket operation is bilinear.*
$(L2)$ $[xx] = 0 \ \ \forall x \in L.$
$(L3)$ $[x[yz]] + [y[zx]] + [z[xy]] = 0 \ \ \forall x, y, z \in L.$

*(L3) is often called the **Jacobi identity**.*

In this talk it will always be assumed that the underlying vector space of a Lie algebra is finite dimensional over F.

Remark I.2 *Similarly to the case of other well known algebraic objects, several fundamental concepts for Lie algebras are defined as follows.*
*A subspace K of L is called a (Lie-)**subalgebra** of L if*

$$[xy] \in K \ \ \forall x, y \in K.$$

*A linear map $\phi : L \to L'$ between two Lie algebras L and L' is called a **homomorphism** if it preserves the bracket, meaning*

$$\phi[xy] = [\phi(x)\phi(y)]$$

*for all $x, y \in L$. Evidently, Im ϕ is then a subalgebra of L'. A homomorphism ϕ is called a **monomorphism** if ker $\phi = 0$, an **epimorphism** if Im $\phi = L'$, and an **isomorphism** if it is both mono- and epimorphism. Two Lie algebras L and L' are called **isomorphic** if there exists an **isomorphism** of Lie algebras $\phi : L \to L'$, i.e. a vector space isomorphism satisfying*

$$\phi[xy] = [\phi(x)\phi(y)].$$

*An **automorphism** of L is an isomorphism of L onto itself. The automorphisms of a Lie algebra L form a group denoted by Aut L.*

To give a first example of a Lie algebra, we introduce the general linear algebra denoted by $\mathfrak{gl}(V)$:

Example I.3 *(general linear algebra $\mathfrak{gl}(V)$)*
Consider a finite-dimensional vector space V over the field F with $\dim V = n$, and let End V denote the space of all endomorphisms of V (i.e., all linear transformations $V \to V$). It can be easily verified that End V forms a ring with the composition of mappings as the ring multiplication and is also a vector space over F with dimension n^2. Introducing a new operation End $V \times$ End $V \to$ End V defined by

$$[x, y] = xy - yx,$$

*called the bracket of x and y, End V becomes a Lie algebra over F. To distinguish the vector space structure of End V from this newly defined Lie algebra structure, we denote End V as a Lie algebra by using the notation $\mathfrak{gl}$**(V)** and call it the **general linear algebra**. This construction for $\mathfrak{gl}(V)$ applies also in case that V is infinite-dimensional. Any subalgebra of $\mathfrak{gl}(V)$ is called a **linear Lie algebra**. Fixing a basis for V one can identify $\mathfrak{gl}(V)$ with the set of all $n \times n$ matrices over F, denoted $\mathfrak{gl}(n, F)$.*

We also want to introduce a couple of subalgebras of $\mathfrak{gl}(V)$:

Example I.4 *(linear Lie algebras)*
To prove that the following examples are indeed Lie algebras is an easy exercise.

*Let $\mathfrak{sl}(V)$ denote the **special linear algebra** which consists of all endomorphisms having trace zero.*

*Let $\mathfrak{o}(V)$ denote the **orthogonal algebra** consisting of all $x \in \mathfrak{gl}(V)$ which fulfil $f(x(v), w) = -f(v, x(w))$ for a non-degenerate bilinear form f on V which is defined by the matrix s. The definition for s depends on the dimension of V. If $\dim V = 2l$, let $s = s_{even}$, if $\dim V = 2l+1$, let $s = s_{odd}$ where s_{even} and s_{odd} are defined as:*

$$s_{even} := \begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix}, \quad s_{odd} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{pmatrix}$$

*Let $\mathfrak{t}(\mathbf{n}, \mathbf{F})$ be the set of all **upper triangular matrices**, $\mathfrak{n}(\mathbf{n}, \mathbf{F})$ the set of all **strictly upper triangular matrices** and $\mathfrak{d}(\mathbf{n}, \mathbf{F})$ the set of all **diagonal matrices** in $\mathfrak{gl}(n, F)$.*

Remark I.5 *We note that*

$$\mathfrak{t}(n, F) = \mathfrak{d}(n, F) + \mathfrak{n}(n, F),$$

where the right hand side is a direct sum of vector spaces. Since $[\mathfrak{d}(n, F), \mathfrak{n}(n, F)] = \mathfrak{n}(n, F)$, it follows that

$$[\mathfrak{t}(n, F), \mathfrak{t}(n, F)] = \mathfrak{n}(n, F).$$

Further important definitions are:

Definition I.6 *(ideal)*
*An **ideal** I of a Lie algebra L is a subspace $I \subset L$ so that it holds true that*

$$x \in L,\ y \in I \Rightarrow [xy] \in I.$$

Example I.7 *(center)*
*Next to the rather trivial examples of 0 (the subspace composed solely of the zero vector) and L itself, a less obvious example for an ideal is the **center** $Z(L)$ of L defined by*

$$Z(L) = \{z \in L \ \ s.t. \ \ [xz] = 0 \ \ \forall x \in L\}.$$

Note that L is abelian precisely when its center is equal to L.

Remark I.8 *A noteworthy observation is that for a homomorphism ϕ, $\ker(\phi)$ is always an ideal of L. Analogous to other algebraic theories, there exists a natural one-to-one correspondence between homomorphisms and ideals: $Ker(\phi)$ is associated with ϕ, and to an ideal I, the canonical map $x \mapsto x + I$ of L onto L/I is assigned, where L/I denotes the quotient algebra defined in the next definition.*

Definition I.9 *(quotient algebra)*
*Let L be a Lie algebra and I a proper nonzero ideal of L. We construct the **quotient algebra**, taking L/I as a quotient of vector spaces for the underlying vector space of the quotient algebra and define a Lie bracket on L/I via*

$$[x + I, y + I] := [xy] + I$$

for all $x, y \in L$. Since this is well defined, L/I becomes a Lie algebra.

To Lie algebras apply the standard homomorphism theorems as well, the proof of which will not be provided as it follows analogously to the case of other algebraic structures.

Proposition I.10 *(homomorphism theorems)*
Let L be a Lie algebra and let I, J be ideals of L.
***(i)** Let $\phi : L \to L'$ be a homomorphism of Lie algebras. Then $L/\ker\phi \cong Im\ \phi$. For any ideal $I \subset \ker\phi$ of L there exists a unique homomorphism $\psi : L/I \to L'$, so that the diagram*

$$\begin{array}{ccc} L & \xrightarrow{\ \phi\ } & L' \\ & \searrow^{\pi} & \uparrow{\scriptstyle\psi} \\ & & L/I \end{array}$$

commutes, where π denotes the canonical projection.
***(ii)** If $I \subset J$, then J/I is an ideal of L/I and there is a natural isomorphism between $(L/I)/(J/I)$ and L/J.*
***(iii)** $(I + J)/J$ is naturally isomorphic to $I/(I \cap J)$.*

For later purposes some more definitions will be needed:

Definition I.11 *(normalizer, centralizer)*
The ***normalizer*** *of a subspace K of L is defined as:*

$$N_L(K) = \{x \in L \mid [xK] \subset K\}$$

In case K is a subalgebra, the Jacobi identity ensures that $N_L(K)$ is again a subalgebra of L, the largest subalgebra that contains K as an ideal. K is called ***self-normalizing*** *if $K = N_L(K)$.*
For a subset $X \subset L$ we define the ***centralizer*** *of X in L by*

$$C_L(X) := \{x \in L \mid [xX] = 0\}.$$

Definition I.12 *(representation) Let V be a vector space over F. A representation of a Lie algebra L on V is a Lie algebra homomorphism*

$$\phi : L \to \mathfrak{gl}(V)$$

While we stipulate that L must be finite-dimensional, it is beneficial to permit V to have arbitrary dimension. The most important example of a representation for our purposes in this talk will be the adjoint representation:

Example I.13 *(adjoint representation)*
The ***adjoint representation*** *of a Lie algebra L is defined by ad : $L \to \mathfrak{gl}(L)$ which sends $x \to [x \cdot]$, so that*

$$ad\ x\ (y) = [xy].$$

Remark I.14 *We observe that the kernel of ad comprises precisely all x in L such that ad x equals 0, meaning [xy] = 0 for every y in L. Hence, the kernel of ad precisely corresponds to the centre of L, so ker ad = $Z(L)$. In case L is simple which implicates $Z(L) = 0$, this means ad : $L \to gl(L)$ is a monomorphism. Consequently, any simple Lie algebra, i.e. a non-abelian, no nonzero proper ideals containing Lie algebra, is isomorphic to a linear Lie algebra.*

II. Nilpotency and Engel's theorem

i. *Nilpotency*

In order to formulate Engel's theorem, we develop the notion of a nilpotent Lie algebra.

Definition II.15 *(lower central series)*
Let L be a Lie algebra. Consider a sequence of ideals of L defined by:

$$L^0 := L$$

$$L^{i+1} := [L\ L^i]$$

The sequence formed by the ideals of the form L^i is called ***lower central series****.*
We have

$$L^1 = [L\ L]$$

and it is not difficult to verify that

$$[L^i\ L^j] \subset L^{i+j}.$$

Definition II.16 *(nilpotent Lie algebra)*
A Lie algebra L is called ***nilpotent*** *if there exists an $n \in \mathbb{N}$, so that*

$$L^n = 0$$

i.e. almost all terms of the lower central series vanish.

Any abelian algebra, for example, is nilpotent. A first observation is the following proposition:

Proposition II.17 *Let L be a Lie algebra.*
(i) *If L is nilpotent, then every subalgebra of L and all images of L under homomorphisms are nilpotent.*
(ii) *If $L/Z(L)$ is nilpotent, then L is nilpotent.*
(iii) *If L is nilpotent and nonzero, then $Z(L) \neq 0$.*

proof:
(i) If K is a subalgebra, then $K^i \subset L^i$ and therefore K is also nilpotent. To see that homomorphic images of L are nilpotent as well, consider an epimorphism $\phi : L \to M$. By induction we obtain $\phi(L^i) = M^i$ and so M is nilpotent too.
(ii) Let n fulfil $(L/Z(L))^n = 0$, then $L^{(n)} \subset Z(L)$ which implies $L^{n+1} = [LL^n] \subset [LZ(L)] = 0$ and so L is nilpotent.
(iii) Since the last non-vanishing term of the lower central series of L is central and nonzero $\Rightarrow Z(L) \neq 0$. □

Definition II.18 *(ad-nilpotent)*
Let L be a Lie algebra and $x \in L$. x is called ***ad-nilpotent*** *if ad x is a nilpotent endomorphism.*

Remark II.19 *We realise that when L is nilpotent then for some $n \in \mathbb{N}$:*

$$ad\ x_1\ ad\ x_2\ \dots\ ad\ x_n(y) = 0$$

for all $x_i, y \in L$. Particularly $(ad\ x)^n = 0$ for all $x \in L$. Therefore all elements of L are ad-nilpotent if L is nilpotent. Engel's theorem will ensure that the converse holds true as well.

Before delving into Engel's theorem, let's examine the following lemma, which will assist us in proving Engel's theorem.

Lemma II.20 *If $x \in \mathfrak{gl}(V)$ is a nilpotent endomorphism, then ad x is nilpotent as well.*

proof (of Lemma II.20):
Let $x \in \mathfrak{gl}(V)$ be nilpotent. We can associate to x two endomorphisms of $End\ V$

$$\lambda_x,\ \rho_x : End\ V \to End\ V$$

defined by $\lambda_x(y) = xy$ and $\rho_x(y) = yx$, called the left and right translation. λ_x and ρ_x obviously commute and since x is nilpotent, λ_x and ρ_x are nilpotent as well. Since in any ring, sums and differences of commuting nilpotent elements are again nilpotent, so is $ad\ x = \lambda_x - \rho_x$. □

ii. *Engel's theorem*

Finally, the necessary grounds are achieved to formulate Engel's theorem.

Theorem II.21 ***(Engel's theorem)***
If all elements of a Lie algebra L are ad-nilpotent, then L is nilpotent.

Engel's theorem will be derived from the subsequent theorem, which holds its own significance.

Theorem II.22 *Let $L \subset \mathfrak{gl}(V)$ be a subalgebra consisting of nilpotent endomorphisms and $V \neq 0$ finite dimensional. Then there exists a nonzero $v \in V$ with $L.v = 0$.*

proof (of Theorem II.22):
This theorem will be proved using induction on dim L. The statement for $dim\ L = 0, 1$ is obvious, if one recalls that a nilpotent linear transformation always has at least one eigenvector to the eigenvalue zero. So now let $dim\ L > 1$ and let $K \neq L$ be a maximal proper subalgebra of L. Lemma II.20 ensures that K acts as a Lie algebra of nilpotent linear transformations via ad on the vector space L, hence also on the vector space L/K, which is nontrivial because K is a proper subalgebra. If $\overline{ad\ K}$ denotes the image of K in $\mathfrak{gl}(V)$, by induction hypothesis there exists a nonzero($x \notin K$) element $x + K \in L/K$ so that

$$\overline{ad\ K}.(x+K) = 0 \Leftrightarrow ad\ K.x = 0$$
$$\Leftrightarrow [Kx] = 0 \Leftrightarrow [yx] = 0 \ \ \forall y \in K.$$

It follows that $K \subsetneq N_L(K) = \{x \in L \mid [xK] \subset K\}$ since $x \notin K$. Because N is a maximal proper subalgebra of L, this directly implies $N_L(K) = L$. Therefore K is an ideal of L. We claim now that $dim\ L/K \leq 1$. To see that suppose $dim\ L/K > 1$ Then L/K would always contain a proper one dimensional subalgebra. Its preimage under the projection map $\pi : L \to L/K$ would be a proper subalgebra properly containing K, which is a contradiction.
Thus K has codimension one. This permitts us to write for any $z \in L \setminus K$:

$$L = K + Fz.$$

Let now $z \in L \setminus K$ and $W = \{v \in V \mid K.v = 0\}$. By the induction hypothesis, W is nonzero. W is stable under L because for $x \in L$, $y \in K$, $w \in W$, it holds that

$$yx.w = xy.w - [xy]w = 0$$

since $[xy] \in K$ because K is an ideal. Therefore z -only acting on W- is a nilpotent endomorphism on W, which implies that it has at least one nontrivial eigenvector $w \in W$ to the eigenvalue zero. This w fulfils now $L.w = 0$. □

Now Engel's theorem follows quite easily.

proof (of Engel's theorem II.21):
Let L be a non-vanishing Lie algebra. We will proof this theorem by induction on *dim L*. With the bilinearity of the bracket it is easy to see that every one dimensional Lie algebra is abelian, hence solvable. Thus the theorem holds for *dim L* = 1. Let now *dim L* be arbitray and the statement be true for every number smaller than *dim L*. Let every element of L be ad-nilpotent, i.e. for all $x \in L$ the endomorphism *ad x* is nilpotent. Since *ad L* consists only of nilpotent elements, and it is a finite dimensional, nontrivial subalgebra of $\mathfrak{gl}(L)$, *ad L* satisfies the preconditions for theorem II.22. Therefore there is a nonzero $y \in L$ for which $(ad\ L).y = 0$.

$\Rightarrow (ad\ L).y = [Ly] = 0$
$\Rightarrow Z(L) \neq 0$ since $y \in Z(L)$
$\Rightarrow L/Z(L) \neq L$

So $L/Z(L)$ has a smaller dimension than L and since all elements of $L/Z(L)$ are ad-nilpotent as well, it follows that $L/Z(L)$ is nilpotent by the induction hypothesis.

$\Rightarrow \exists n \in \mathbb{N}\ s.t.\ (L/Z(L))^n = 0$
$\Rightarrow L^n \subset Z(L)$
$\Rightarrow L^{n+1} = [LL^n] \subset [LZ(L)] = 0$
$\Rightarrow L$ is nilpotent

□

A corollary of significance arises from theorem II.22 that is actually equivalent to theorem II.22 itself. To articulate this corollary, the following definition will be needed.

Definition II.23 *(flag)*
Let V be a finite dimensional vector space of $dim\ V = n$. A ***flag*** *in V is a chain of subspaces*

$$0 = V_0 \subset V_1 \subset \ldots \subset V_n = V$$

where $dimV_i = i$. $x \in End\ V$ is said to ***stabilise*** *this flag if for all i it holds true that:*

$$x(V_i) \subset V_i.$$

We conclude this section now with the promised corollary.

Corollary II.24 *Let L be a subalgebra of $\mathfrak{gl}(V)$ and $V \neq 0$ finite dimensional. If L consists of nilpotent endomorphisms, then there exists a flag (V_i) in V which is stabilised by L and which fulfils $L.V_i \subset V_{i-1}$ for all i.*
Put differently, the matrices of L, expressed in a suitable basis of V, all lie in $\mathfrak{n}(n, F)$, i.e. are all strictly upper triangular.

III. Solvability and Lie´s theorem

i. *Solvability*

Solvability will be defined in a manner very similar to nilpotency, given their close relationship. In fact, solvability will turn out to be a generalisation of nilpotency.

Definition III.25 *(derived series)*
Let L be a Lie algebra. Consider a sequence of ideals of L defined by:

$$L^{(0)} := L$$
$$L^{(i+1)} := [L^{(i)}\ L^{(i)}]$$

The sequence formed by the ideals of the form $L^{(i)}$ is called ***derived series****.*

Definition III.26 *(solvable Lie algebra)*
A Lie algebra L is called ***solvable*** *if there exists an $n \in \mathbb{N}$ so that*

$$L^{(n)} = 0$$

i.e. almost all terms of the derived series vanish.

Every nilpotent algebra is solvable due to the inclusion $L^{(i)} \subset L^i$. However, the contrary is not necessarily true.

Proposition III.27 *Let L be a Lie algebra*
(i) *Every subalgebra of L and all images of L under homomorphisms are solvable if L is solvable.*
(ii) *If there exists a solvable ideal I of L so that L/I is solvable, then L is also solvable.*
(iii) *If I, J are solvable ideals of L, then $I + J$ is solvable as well.*

proof:
(i) The proof works analogously to the proof of (i) of proposition II.17.
(ii) Let n fulfil $(L/I)^{(n)} = 0$ and let $\pi : L \to L/I$

be the canonical homomorphism. Using (i) it follows that $\pi(L^{(n)}) = 0 \Leftrightarrow L^{(n)} \subset I = ker\ \pi$. With m fulfilling $I^{(m)} = 0$ and $(L^{(i)})^{(j)} = L^{(i+j)}$ we can conclude $L^{(n+m)} = 0$

(iii) $(I + J)/J \cong I/(I \cap J)$ follows from (iii) of the homomorphism theorems I.10. Since $I/(I \cap J)$ is a homomorphic image of I it is solvable and so is $(I + J)/J$. By assumption also J is solvable and with (ii) it follows that $(I + J)/J$ is solvable. □

ii. *Lie's theorem*

For this last section the underlying field F will be assumed to be algebraically closed and have $char\ F = 0$. The main difficulty lies in proving the following theorem which lets Lie's theorem follow as a corollary.

Theorem III.28 *If $L \subset \mathfrak{gl}(V)$ is a solvable subalgebra and $V \neq 0$ is finite dimensional, then V contains a common eigenvector for all endomorphisms in L.*

sketch of the proof:
The strategy employed to prove this theorem closely resembles the approach used in the proof of Theorem II.22. It involves induction on the dimension of L, with the statement for $dim\ L = 0$ being trivial. Then the idea is to proceed via the following steps:
(1) find an ideal K of codimension one
(2) verify that for K there exist common eigenvectors (by induction)
(3) show that a space W of such eigenvectors is stabilised by L
(4) for a $z \in L$ with $L = K + Fz$, find an eigenvector of z in W.
This ensures then that there is a common eigenvector for all endomorphisms in L.

Corollary III.29 *(Lie's theorem)*
If $L \subset \mathfrak{gl}(V)$ is a solvable subalgebra and V is finite dimensional, then there exists a flag of V which is stabilised by L. That is to say there exists a basis V for which the corresponding matrices of L all lie in $\mathfrak{t}(n, F)$, i.e. are all upper triangular.

proof:
To deduct Lie's theorem as a corollary from the previous theorem we use again induction on the dimension of L. The statement is obviously true for $dim\ L = 0$, so lets assume $dim\ L > 0$ and the validity of the statement for all $m < dim\ L$. Using theorem III.28 there has to exist a nonzero $v \in V$ that is a common eigenvector of L. Let now $V_1 := span_F(v)$. Clearly $dim\ V_1 = 1$, so $dim\ V/V_1 < dim\ L$. By induction hypothesis V/V_1 permits therefore a flag which is stabilised by L, say

$$0 = W_0 \subset W_1 \subset \ldots \subset W_{n-1} = V/V_1.$$

With the preimage of this flag under the canonical projection of V onto V/V_1, one obtains a flag of V that is stabilised by L. □

Now that we have established Lie's theorem, there are two additional interesting results that can be derived as corollaries from the previous theorems.

Corollary III.30 *(of Lie's theorem)*
If L is solvable, then there exists a chain of ideals

$$0 = L_0 \subset L_1 \subset \ldots \subset L_n = L$$

with $dim\ L_i = i$.

proof:
Consider the adjoint representation $ad : L \to \mathfrak{gl}(L)$. Since it is a finite dimensional representation, proposition III.27 (i) ensures that $ad\ L$ is solvable too. Due to Lie's theorem it therefore stabilises some flag of L, which is then a chain of ideals in L. □

Corollary III.31 *(of Engel's and Lie's theorem)*
If L is solvable, then $ad_L\ x$ is nilpotent for all $x \in [LL]$. Particularly, $[LL]$ is nilpotent.

proof:
By the previous corollary we know that there exists a flag $(L_i)_i$ of L consisting of ideals. Let $(x_1, \ldots, x_n)$ be a basis of L so that $span(x_1, \ldots, x_i) = L_i$. In this basis, the matrices corresponding to $ad\ L$ are all upper triangular, hence in $\mathfrak{t}(n, F)$. Since $[\mathfrak{t}(n, F)\mathfrak{t}(n, F)] = \mathfrak{n}(n, F)$, it follows that $ad\ [LL] = [ad\ L, ad\ L] \subset \mathfrak{n}(n, F)$. Therefore $ad\ x$ is nilpotent for all $x \in [LL]$ and so by Engel'theorem $[LL]$ is nilpotent. □

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