8. Root space Decomposition (p. 35 - 45)

$\longrightarrow$ $0 \neq L$ semisimple $\left| \Leftrightarrow [LL] = L \Leftrightarrow Z(L) = 0 \right|$

$\rightsquigarrow$ keep $L = sl(2, F) = \langle \underset{e_{12}}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \underset{e_{21}}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \underset{h_1}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \rangle$

as an example in mind

If $\forall x \in L$: x nilp $\overset{\text{Engel's Thm.}}{\Longrightarrow}$ L nilp. . If L is not nilp $\Rightarrow \exists y \in L$ with $y = y_s + y_n$ st. $y_s \neq 0$ $\Rightarrow \exists\, 0 \neq S \subseteq L$: $s \in S$: s is semisimple

$\longrightarrow$ S is called toral subalg.

Lemma: $\forall H \subseteq L$, H is a toral subalg. $\Rightarrow$ H is abelian

$\boxed{\text{proof}}$: have to show: $ad_H x = 0$ $\forall x \in T$ (LA, eigenval. + -vec. leading to a contradiction) ∎

- fix max. toral subalg. H: (will see that H is a Cartan subalg.)

  $\overset{H\text{ ab.}}{\Longrightarrow}$ $ad_L H$ com. fam. of semisimple endos.

  $\Rightarrow$ $ad_L H$ is simult. diag.

  $\Rightarrow$ L is the direct sum of

  $L_\alpha := \{x \in L \mid [hx] = \alpha(h) x \ \forall h \in H\}$ where $\alpha \in H^*$ ranges over H^*

  where $L_0 = C_L(H)$ (wanna show: $H = C_L(H)$)

Def: $\Phi := \{\alpha \in H^* \mid L_\alpha \neq 0\}$ it's elem. are called roots of L relativ to H

$\longrightarrow$ root sp. decomp. (Cartan Decomp.): $L = \underset{L_0}{\underset{\shortparallel}{C_L(H)}} \oplus \bigsqcup_{\alpha \in \Phi} L_\alpha$ $(*)$

(*) for $sl(n,F)$ given by the standard basis

$\rightarrow sl(2,F) = \langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rangle \oplus \langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rangle \sqcup \langle \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \rangle$

$sl(3,F) = \langle h_1, h_2 \rangle \oplus \bigsqcup_{i \neq j} \langle e_{ij} \rangle$

Prop: For all $\alpha, \beta \in H^*$, $[L_\alpha, L_\beta] \subseteq [L_{\alpha+\beta}]$. If $x \in L_\alpha$, $\alpha \neq 0$, then $\operatorname{ad} x$ is nilp.. If $\alpha, \beta \in H^*$, $\alpha + \beta \neq 0 \Rightarrow L_\alpha \perp_{\kappa_L} L_\beta$

$\boxed{\text{proof}}$ Jac. id.: $x \in L_\alpha$, $y \in L_{-\alpha}$, $h \in H$:

$\operatorname{ad} h([x,y]) = [h[x,y]] = [[hx],y] + [x[hy]] = \alpha(h)[xy] + \beta(h)[xy]$

$= (\alpha+\beta)(h)[xy]$

$\rightsquigarrow$ find an h for which $(\alpha+\beta)(h) = 0 \xrightarrow{\text{calc.}} \kappa(x,y) = 0$ ■

Cor.: $\kappa|_{L_0 = C_L(H)}$ is nondegen.

$\boxed{\text{proof}}$: we know κ_L is nondegen. (L semisimple) + $L_0 \perp_{\kappa_L} L_\alpha$ for all $\alpha \in \Phi$

if $z \in L_0$ is orth. to L_0 $\Rightarrow \kappa(z, L) = 0 \overset{\kappa \text{ nond.}}{\Rightarrow} z = 0$ ■

// LA (i): x, y com. endos of fin. dim. v.sp., y nilp. $\Rightarrow$ xy nilp. + $Tr(xy) = 0$

Prop.: H max. tor. subalg. $\Rightarrow H = C_L(H) =: C$

$\boxed{\text{proof}}$: ① C contains the nilp. + semis. parts of its elem. ($\operatorname{ad} x(H) = 0$ for $x \in C$)

② all semis. elem. of C lie in H $\left(H + F_x \text{ toral}, x \text{ sem.s.} \xrightarrow{\text{max.}} H = \overbrace{H + F_x}^{\text{con.}} \right)$

③ $\kappa|_H$ is nondegen. (straightforward)

④ C is nilp. (Engels) + LA (i)

⑤ $H \cap [CC]$ (κ assoc.)

⑥ C is abelian (assume $[C,C] \neq 0$)

⑦ $C = H$ (assume its not $\longrightarrow$ ↯ $\kappa|_C$ nondegen.

■

Cor.: $\kappa|_H$ is nondegen.

■

$\rightarrow$ cor. allows us to identify H with H^*:

$\phi \in H^*$ corr. the (unique) $t_\phi \in H$ satisfying $\phi(h) = \kappa(t_\alpha, h)$ $\forall h \in H$

$\rightsquigarrow$ Φ corr. to the subset $\{t_\alpha \mid \alpha \in \Phi\}$ of H

Orthog. pplies

Prop.: (a) Φ spans H^*

(b) If $\alpha \in \Phi \Rightarrow -\alpha \in \Phi$

(c) Let $\alpha \in \Phi$, $x \in L_\alpha$, $y \in L_{-\alpha}$. Then $[xy] = \kappa(x,y)\, t_\alpha$

(d) If $\alpha \in \Phi$, then $[L_\alpha, L_{-\alpha}]$ is one dim. with basis t_α

(e) $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$ for $\alpha \in \Phi$

(f) If $\alpha \in \Phi$, $x_\alpha \neq 0$ elem. of $L_\alpha \Rightarrow \exists y_\alpha \in L_{-\alpha}$: $x_\alpha, y_\alpha, h_\alpha = [x_\alpha y_\alpha]$ span a 3-dim. subalg $S \cong sl(2,F)$ via $x_\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $y_\alpha \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h_\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(g) $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$; $h_\alpha = -h_{-\alpha}$

$\boxed{\text{proof}}$: (a) ass. Φ fails to span H^* $\overset{\text{by duality}}{\Rightarrow}$ $\exists 0 \neq h \in H$ st $\alpha(h) = 0 \ \forall \alpha \in \Phi$

$\Rightarrow [h, L_\alpha] = 0 \ \forall \alpha \in \Phi \overset{H \text{ ab.}}{+} [h H] = 0 \Rightarrow [h, L] = 0 \Leftrightarrow h \in Z(L) = 0$ ↯

(b) $-\alpha \notin \Phi \overset{\text{Def.}}{\Rightarrow} L_{-\alpha} = 0 \Rightarrow \kappa(L_\alpha, L_\beta) = 0 \ \forall \beta \in H^*$

$\Rightarrow \kappa(\underset{0}{\overset{\neq}{}}L_\alpha, L) = 0$ ↯ κ nondegen. radical $(\kappa) = 0$

(c) $\alpha \in \Phi$, $x \in L_\alpha$, $y \in L_{-\alpha}$ $\Rightarrow [x, y] = \kappa(x, y)\, t_\alpha$

$\rightarrow$ $h \in H$ arbitr.: $\kappa(h, [x, y]) = \kappa([h x], y) = \kappa(\alpha(h) x, y)$

$= \alpha(h)\, \kappa(x, y)$

$= \kappa(t_\alpha, h)\, \kappa(x, y)$

$= \kappa(\kappa(x, y) t_\alpha, h)$

$= \kappa(h, \kappa(x, y) t_\alpha)$

bc. $\kappa|_H$ is nondegen. $\Rightarrow [x, y] - \kappa(x, y) t_\alpha = 0$

$\Leftrightarrow [x, y] = \kappa(x, y)\, t_\alpha$

$\overset{(c)}{\Rightarrow}$ (d) $\langle t_\alpha \rangle = [L_\alpha L_{-\alpha}]$ (1-dim.) if $[L_\alpha L_{-\alpha}] \neq 0$

$0 \neq x \in L_\alpha$. If $\kappa(x, L_{-\alpha}) = 0 \Rightarrow \kappa(x, L) = 0$ ↯ κ nondegen.

$\Rightarrow \exists 0 \neq y \in L_{-\alpha}$: $\kappa(x, y) \neq 0 \Rightarrow [x, y] = 0$

e) $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0 \quad \forall \alpha \in \Phi$

$\to$ supp. $\alpha(t_\alpha) = 0$ st. $[t_\alpha x] = 0 = [t_\alpha y] \quad \forall x \in L_\alpha, y \in L_{-\alpha}$

$\overset{(d)}{\leadsto} \exists x \in L_\alpha, y \in L_{-\alpha} : \kappa(x,y) \neq 0 \to$ assume $\kappa(x,y) = 1 \Rightarrow [x,y] = t_\alpha$

$S := \langle x, y, t_\alpha \rangle$ is a 3-dim. solvable subalg.

with $S \cong \mathrm{ad}_L S \subset gl(L) \quad ([S,S] = \langle t_\alpha \rangle =: S' \to [S', S'] = 0)$

$\leadsto$ in part. $\mathrm{ad}_L s$ is nilp. $\forall s \in [S,S]$

$\Rightarrow \mathrm{ad}_L t_\alpha$ is both nilp. & semisimple, ie. $\mathrm{ad}_L t_\alpha = 0$

$\Rightarrow t_\alpha \in Z(L)$ ↯ (Why?)

(f) given $0 \neq x_\alpha \in L_\alpha$ find $y_\alpha \in L_{-\alpha}$ st. $\kappa(x_\alpha, y_\alpha) = \dfrac{2}{\kappa(t_\alpha, t_\alpha)}$

possible bc. of (e) and $\kappa(x_\alpha, L_{-\alpha}) \neq 0$

$h_\alpha := 2t_\alpha / \kappa(t_\alpha, t_\alpha) \Rightarrow [x_\alpha y_\alpha] = h_\alpha$

moreover $[h_\alpha x_\alpha] = \dfrac{2}{\alpha(t_\alpha)} [t_\alpha x_\alpha] = 2x_\alpha$

$[h_\alpha y_\alpha] = -2y_\alpha$

$\to S \cong sl(2, F)$

(g) t_α is def. as $\kappa(t_\alpha, h) = \alpha(h) \quad (h \in H) \Rightarrow t_\alpha = -t_{-\alpha}$

$\Rightarrow h_\alpha = -h_{-\alpha}$

Integrality pptics

(each pair)
for $\alpha, -\alpha \in \Phi$ we get $S_\alpha = \langle x_\alpha, y_\alpha, h_\alpha \rangle \cong sl(2,F)$

Weyls Thm. + Thm 7.2 $\longrightarrow$ complete discription of all S_α-modules

...

Sumarize:

Prop.: (a) $\alpha \in \Phi \Rightarrow \dim L_\alpha = 1$ and in part. $S_\alpha = L_\alpha + L_{-\alpha} + H_\alpha$ where $H_\alpha = [L_\alpha L_{-\alpha}]$

$\rightarrow$ also for $0 \neq x_\alpha \in L_\alpha \ \exists! \ y_\alpha \in L_{-\alpha} : [x_\alpha y_\alpha] = h_\alpha$

(b) $\alpha \in \Phi \Rightarrow \pm\alpha \in \Phi$ only scalar mult. of α in Φ

(c) $\alpha, \beta \in \Phi \Rightarrow \beta(h_\alpha) \in \mathbb{Z}$ and $\beta - \beta(h_\alpha)\alpha \in \Phi$

(d) $\alpha, \beta, \alpha+\beta \in \Phi : [L_\alpha, L_\beta] = L_{\alpha+\beta}$

(e) $\alpha, \beta \in \Phi$, $\beta \neq \pm\alpha$. Let q, r be the greatest int., st. $\beta - r\alpha, \beta + q\alpha$ are roots

$\Rightarrow \beta + i\alpha \in \Phi \ \forall -r \leq i \leq q$ & $\beta(h_\alpha) = r - q$

(f) L is gen. (as a Lie alg.) by the L_α

Rationality props

$\rightsquigarrow$ L semisimple Lie alg. $/F$ (alg. closed, char$(F)=0$), H max. tor. subalg.

$\Phi \subseteq H^*$ set of roots rel. to H, $L = H + \coprod_{\alpha\in\Phi} L_\alpha$

$\rightarrow$ basis $\alpha_1,\dots,\alpha_\ell \in H^*$ for H^*

each $\beta\in\Phi$: $\beta = \sum_{i=1}^{\ell} c_i\alpha_i$, $c_i\in F$ $\longrightarrow$ c_i is even an elem. of $\mathbb{Q}$ (calc.)

$\rightarrow$ $\mathbb{Q}$-subsp. $E_{\mathbb{Q}}$ of H^* spanned by all roots has $\mathbb{Q}$-dim $\ell = \dim_F H^*$

def. $\kappa^*(\gamma,\delta) = \kappa(t_\gamma, t_\delta)$ with $\gamma,\delta\in H^*$ (transfer from H to H^*)

$\rightsquigarrow$ κ^* is pos. def. on $E_{\mathbb{Q}}$

Let E be a real v. sp. obtained by $E := \mathbb{R}\otimes_{\mathbb{Q}} E_{\mathbb{Q}}$ ext. the base field $\mathbb{Q}$ to $\mathbb{R}$

$\rightsquigarrow$ κ^* ext. can. to E + stays pos. def.

$\rightarrow$ E is an eucl. sp., Φ contains a basis of E & $\dim_{\mathbb{R}} E = \ell$

sum. up:

Thm.: L, H, Φ, E as above

(a) Φ spans E, $0\notin\Phi$

(b) if $\alpha\in\Phi \Rightarrow \lambda\alpha\in\Phi$ for $\lambda=\pm 1$

(c) if $\alpha,\beta\in\Phi$, then $\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \Phi$

(d) if $\alpha,\beta\in\Phi \Rightarrow \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$

$\rightsquigarrow$ import. for chp. III

$\rightarrow$ corr. $(L,H) \xrightarrow{1:1} (\Phi, E)$

9.1. Reflections in a euclidean space

→ fixed eucl. v. sp., i.e. fin. dim. v.sp. over $\mathbb{R}$ with a pos. def. bilf. (α, β)

• $\alpha \in E$ nonzero determines a refl. σ_α with reflecting hypp. $P_\alpha := \{\beta \in E \mid (\beta, \alpha) = 0\}$

→ def. $\sigma_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \alpha$ (works, bc. $\sigma_\alpha(\alpha) = -\alpha$ and $\sigma_\alpha(\beta) = \beta \ \forall \beta \in P_\alpha$)

⇝ def. $\langle \beta, \alpha \rangle := \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$

Lem.: Let Φ be a fin. set, that spans E. Supp. all refl. σ_α ($\alpha \in \Phi$) leave Φ inv.. If $\sigma \in GL(E)$ leaves Φ inv., fixes pointw. a hypp. P of E and sends some nonz. $\alpha \in \Phi$ to its negativ, then $\sigma = \sigma_\alpha$ (and $P = P_\alpha$)

Pf: Let $\tau = \sigma\sigma_\alpha$ ($= \sigma\sigma_\alpha^{-1}$). Then $\tau(\Phi) = \Phi$ and $\tau(\alpha) = \alpha$ and τ acts as $id_{\mathbb{R}\alpha}$ as well as $id_{E/\mathbb{R}\alpha}$, so all eig. val. of τ are 1 and the min pol. of τ divides $(T-1)^\ell$ ($\ell = \dim E$). On the other hand, since Φ is fin., not all vec. $\beta, \tau(\beta), \tau^2(\beta), \ldots, \tau^k(\beta)$ ($\beta \in \Phi$, $k \geq |\Phi|$) can be distinct, so some power of τ fixes β. Chose k large enough (lowest common mult.?) so that τ^k fixes all $\beta \in \Phi$. Bc. Φ spans E, this forces $\tau^k = 1$, so the min. pol. of τ divides $T^k - 1$

$\Rightarrow$ τ has min. pol. $T - 1 = \gcd(T^k - 1, (T-1)^\ell)$, ie. $\tau = 1$ ∎

9.2 Root System

(reduced)*

A subset Φ of E is called a root system in E if the foll. axioms are sat.:

(R1) Φ fin., spans E and $0 \notin \Phi$

(R2) If $\alpha \in \Phi$, then the only mult. of α in Φ are $\pm\alpha$ (*)

(R3) If $\alpha \in \Phi$, then the refl. σ_α leaves Φ inv.

(R4) If $\alpha, \beta \in \Phi$, then $\langle \beta, \alpha \rangle \in \mathbb{Z}$

$\rightsquigarrow$ also called reduced root syst. (without (R2) it would still be a root syst.)

Let Φ denote a root syst. in E. Denote by $\mathcal{W}$ the subgrp. of $GL(E)$ gen. by the refl. σ_α ($\alpha \in \Phi$). By (R3) $\mathcal{W}$ permutes the set Φ which by (R1) spans E. This allows us to identify $\mathcal{W}$ with a subgrp. of the sym. grp. on Φ; in part. $\mathcal{W}$ is fin.

$\rightarrow$ $\mathcal{W}$ is called Weyl group of Φ

$\rightsquigarrow$ certain autom. of E act on $\mathcal{W}$ by conjugation.

Lem.: Let Φ be a root syst. in E, with Weyl grp. $\mathcal{W}$. If $\sigma \in GL(E)$ leaves Φ inv., then $\sigma \sigma_\alpha \sigma^{-1} = \sigma_{\sigma(\alpha)}$ for all $\alpha \in \Phi$ and $\langle \beta, \alpha \rangle = \langle \sigma(\beta), \sigma(\alpha) \rangle$ for all $\alpha, \beta \in \Phi$.

$\boxed{\text{pf}}$: $\sigma \sigma_\alpha \sigma^{-1}(\sigma(\beta)) = \sigma \sigma_\alpha(\beta) \in \Phi$. But this equals $\sigma(\beta - \langle \beta, \alpha \rangle \alpha) = \sigma(\beta) - \langle \beta, \alpha \rangle \sigma(\alpha)$. Since $\sigma(\beta)$ runs over

Φ as β runs over Φ, we conclude that $\sigma\sigma_\alpha\sigma^{-1}$ leaves Φ inv., while fixing pointw. the hypp. $\sigma(P_\alpha)$ and sending $\sigma(\alpha)$ to $-\sigma(\alpha)$. By Lem. 9.1, $\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$. But then, comparing the equations above with the equation $\sigma_{\sigma(\alpha)}(\sigma(\beta)) = \sigma(\beta) - \langle\sigma(\beta), \sigma(\alpha)\rangle\, \sigma(\alpha)$ we also get the sec. assertion ▪

$\rightsquigarrow$ natural notion of isom. betw. root syst. Φ and Φ' in resp. eucl. sp. E and E':

Call (Φ, E) and (Φ', E') isom. if there ex. vec. sp. isom. $\phi: E \to E'$ sending Φ to Φ' s.t. $\langle\phi(\alpha), \phi(\beta)\rangle = \langle\beta, \alpha\rangle$ for each pair of roots $\alpha, \beta \in \Phi$.

$\Rightarrow \quad \sigma_{\phi(\alpha)}(\phi(\beta)) = \phi(\sigma_\alpha(\beta))$.

Therefore an isom. of root syst. induces a natural isom $\sigma \longmapsto \phi\sigma\phi^{-1}$ of Weyl grps.

$\to$ an autom. of Φ is the same as an autom. of E leaving Φ inv.

$\to$ in part.: can view $\mathcal{W}$ as a subgrp. of $\mathrm{Aut}\,\Phi$

useful to work also with $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$

$\to$ $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$ the dual (or inverse) of Φ

$\to$ is also a root syst. with Weyl grp. isom. to $\mathcal{W}$

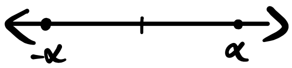
(Lie alg.: $\alpha \leftrightarrow t_\alpha$ and $\alpha^\vee \leftrightarrow h_\alpha$)

9.3 Examples

Call $\ell = \dim E$ the rank of the root system Φ.

$\rightsquigarrow$ for $\ell \leq 2$: we can describe Φ by drawing

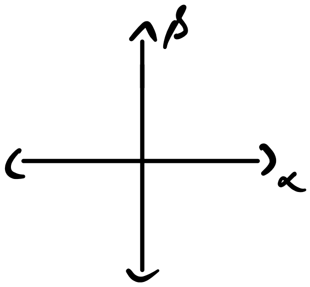
$\ell = 1$



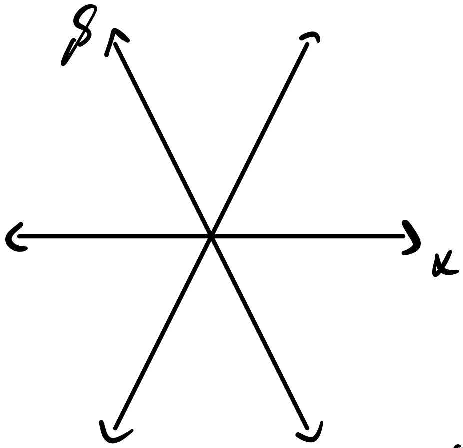
(A_1) (bc. of (R2))

root syst. ($\mathcal{W}$ of order 2) (Lie alg. theory: belongs to $sl(2,F)$)

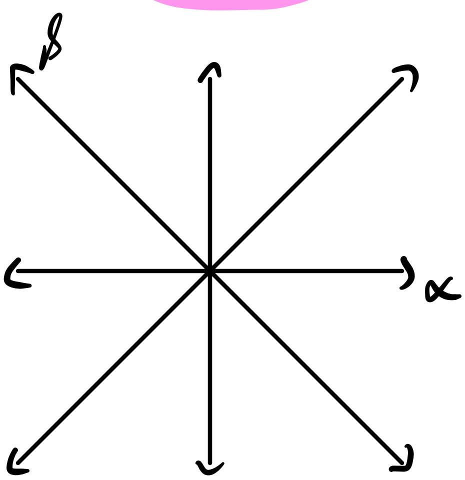
$\ell = 2$



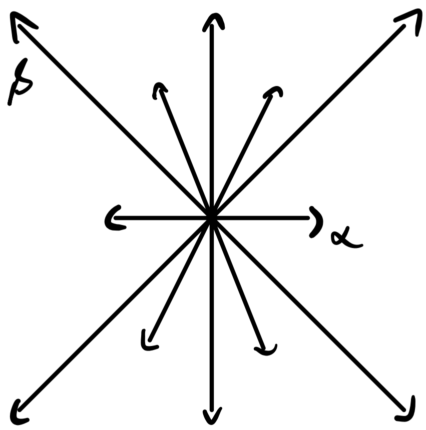
$(A_1 \times A_1)$



(A_2)



(B_2)



(G_2)

9.4 Pairs of Roots

(R4) limits the possible angles occ. betw. pairs of roots

→ cosine of an angle θ betw. vec. α, β is given by

$\|\alpha\| \|\beta\| \cos\theta = (\alpha, \beta)$

$$\Rightarrow \langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)} = 2 \frac{\|\beta\|}{\|\alpha\|} \cos\theta$$

$$\Rightarrow \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 4 \cos^2\theta \geq 0$$

⇝ notice: $0 \leq \cos\theta \leq 1$ and $\langle \beta, \alpha \rangle, \langle \alpha, \beta \rangle$ have same sign

the foll. pos. are the only ones for $\alpha \neq \pm\beta$ and $\|\beta\| \geq \|\alpha\|$

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	θ	$\|\beta\|^2 / \|\alpha\|^2$
0	0	$\pi/2$	—
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3