Representations of $\mathfrak{sl}_2(\mathbb{C})$

Recall: $\mathfrak{g}$ a Lie algebra, V v. space

Def.: 1) A $\mathfrak{g}$-module structure is given by a linear map:

$$\pi : \mathfrak{g} \longrightarrow \mathrm{End}(V), \text{ satisfying: } \pi([X,Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X)$$

(V, π) is called $\mathfrak{g}$-module

2) $\mathfrak{g}$-module irreducible $:\Leftrightarrow$ only π-invariant subspaces are V and (0)

$\mathfrak{sl}_2(\mathbb{C})$:
- special linear Lie algebra
- 2×2-matrices with trace 0 and complex entries

Basis:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Bracket relations:

$$[H,X] = 2X \quad [H,Y] = -2Y \quad [X,Y] = H$$

Rk:
- X, Y nilpotent
- $H \cdot \mathbb{C} = \mathfrak{h}$ Cartan subalgebra ($\Rightarrow$ abelian, elements semisimple)

Def.: V finite $\underline{sl}_2$-module, $\lambda \in \mathbb{C}$

- V_λ denotes eigenspace of H in V corresponding to λ

  ↳ n-case: V_λ space of simultaneous e.v. to all $H \in \mathfrak{h}$
  called <u>weight-space</u>

- $v \in V_\lambda$ has <u>weight</u> λ

Prop.: i) $\bigoplus_{\lambda \in \mathbb{C}} V_\lambda = V$ (for inf. dim only direct sum, not $= V$)

ii) for $v \in V_\lambda$
- $Xv \in V_{\lambda+2}$
- $Yv \in V_{\lambda-2}$

proof: i) $\mathbb{C}$ algebraically closed $\Rightarrow \lambda$ distinct $\overset{LA1}{\Rightarrow}$ V is direct sum of V_λ

ii) Calculation: $v \in V_\lambda$

$HXv = [H,X]v + XHv = 2Xv + X\lambda v = (\lambda+2)Xv \Rightarrow Xv \in V_{\lambda+2}$

(similar for Y)

$\square$

Observation: action of X raises weight
Y lowers weight

Def.: $e \in V \setminus \{0\}$ is called "primitive element" of weight λ
$:\Leftrightarrow Xe = 0$ and $He = \lambda e$

Prop.: V $\underline{sl}_2$-module $V \neq (0)$, $\dim V < \infty$, then V contains a primitive element

proof: i) Lie's Theorem

or ii) Let v be e.vect. for H: The sequence $v, Xv, X^2v, \dots$ terminates because V is fin. dim., so there is $V_{\lambda+n} = (0)$

Choose last non-zero term $\Rightarrow X^i(v)$ is primitive element

Submodules generated by primitive elements

V $\underset{\sim}{sl_2}$-module, $e \in V$ prim. elt. of weight λ

Action of H, X, Y on V_λ's:

$$\begin{tikzcd}
\cdots \arrow[r, dashed] \arrow[r, bend left] & V_{\lambda-4} \arrow[loop below, "H"] \arrow[r, bend left, "X"] & V_{\lambda-2} \arrow[l, "Y"'] \arrow[loop below, "H"] \arrow[r, bend left, "X"] & V_\lambda \arrow[l, "Y"'] \arrow[loop below, "H"]
\end{tikzcd}$$

Observation: Y can „move" prim elt. e through all V_λ's $\Rightarrow \{e, Ye, Y^2e, \dots\}$ spans V

We define a sequence, which will help us prove this

$$e_n = Y^n e \frac{1}{n!} \qquad (e_{-1} = 0)$$

Consider action of std. basis:

(1) $He_n = (\lambda - 2n) e_n$

(2) $Ye_n = (n+1) e_{n+1}$ → lowering weight

(3) $Xe_n = (\lambda - n + 1) e_{n-1}$ → raising weight

proof: (1) $e \in V_\lambda \Rightarrow e_n \in V_{\lambda-2n}$

(2) $Ye_n = Y Y^n e \frac{1}{n!} = Y^{n+1} e \frac{(n+1)}{(n+1)!} = (n+1) e_{n+1}$

(3) Induction of n

i) $n=0$: $e_{n-1} = 0$

only if time permits

↳ $n \to n+1$: $(n+1) X e_{n+1} = X Y e_n = [X,Y] e_n + YXe_n = He_n + Y((\lambda-n+1)e_{n-1})$

$= (\lambda - 2n) e_n + n(\lambda - n + 1) e_n = (n+1)(\lambda - n) e_n$

$\Rightarrow X e_{n+1} = (\lambda - n) e_n$

$\Rightarrow X e_n = (\lambda - n + 1) e_{n-1}$ $\square$

Cor.: $\lambda = m \in \mathbb{N}$, $e_1, \dots, e_m$ lin indep. and $e_i = 0 \ \forall i > m$

proof: lin. indep, due to distinct weights

V is fin. dim $\Rightarrow \exists m \in \mathbb{N}$ s.t. $V_{\lambda + m} \neq (0)$ and $V_{\lambda + (m+1)} = (0) \Rightarrow e_i = 0 \ \forall i > m$

$(3) \Rightarrow X e_{m+1} = 0 = (\lambda - m) e_m \Rightarrow \lambda = m$

□

Let $W \subseteq V$ with $\mathcal{B}_W = \{e_0, \dots, e_m\}$

Cor.: i) W is stable under $\underline{sl_2}$

ii) W is an irreducible $\underline{sl_2}$-module

proof: i) Formulas show: $H(W) \subseteq W$

$X(W) \subseteq W$

$Y(W) \subseteq W$

ii) Let $W' \subseteq W$

$(1) \Rightarrow$ e. val. of H in W are $m, m-2, \dots, -m$ with multip. 1

$W' \subseteq W \Rightarrow \mathcal{B}_{W'} \subseteq \mathcal{B}_W$ is basis for $W' \Rightarrow e_i \ (0 \leq i \leq m) \in W'$

(2), (3) permit raising and lowering weight $\Rightarrow \{e_0, \dots, e_m\} \subseteq W'$

$\Rightarrow W' = W$, W irreducible

□

Classifying W_m-modules

Let W_m be a v. space, $\mathcal{B} = \{e_0, \dots, e_m\} \Rightarrow \dim W_m = m+1$

and endomorphisms : h, x, y on W_m, s.t. :

- $h e_n = (m-2n) e_n$, $y e_n = (n+1) e_{n+1}$, $x e_n = (m-n+1) e_{n-1}$
- $h x e_n - x h e_n = 2 x e_n$, $h y e_n - y h e_n = -2 y e_n$, $x y e_n - y x e_n = h e_n$

$\Rightarrow h, x, y$ induce a $\underline{sl_2}$-module structure on W_m

Theorem: Let V be an irred. $\underline{sl_2}$-module , $\dim V = m+1$

i) W_m is irreducible

ii) $V \cong W_m$

proof: i) follows from last Cor. and W_m is generated by images of e_0 with weight m

ii)
- V contains prim. elt. v of weight w
- $w \in \mathbb{N}$ and $W' \subseteq V$ generated by v has $\dim W' = w+1$
- V irred $\Rightarrow W' = V$ and $w = m$
- Applying formulae shows $\Rightarrow V \cong W_m$ $\square$

Structure of the modules

Let V be a $\underline{sl_2}$-module of $\dim V < \infty$

Thm.: $V \cong \bigoplus W_m$

proof: Weyl's theorem: every fin. dim linear repr. of semi-simple Lie algebra is completely reducible

So V is isomorphic to a sum of irred. modules V_i

$\Rightarrow$ each $V_i \cong W_{m_i}$ for some m_i

$\Rightarrow V \cong \bigoplus V_i \cong \bigoplus W_m$

Thrm.: i) The induced endomorphism of H is diagonizable with integer e.val.'s and if $\pm n$ is e.val, so are $n-2, n-4, \dots, -n$

ii) If n is a non-zero integer: $Y_n : V_n \longrightarrow V_{-n}$, $X_n : V_{-n} \longrightarrow V_n$ are isomorphisms (and V_n, V_{-n} have same dim)

proof: both (i) and (ii) follow from earlier theorems (assume V is W_m ...) $\square$

Important example:

Thrm. W_2 is isomorphic to $\mathrm{Sym}^2(\mathbb{C}^2)$

proof: i) W_0 is the trivial module

ii) W_1 : $(W_1, \{H, X, Y\})$, $B_{\mathbb{C}^2} = \{x, y\}$

$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ has eigenvalues $\lambda = 1$ and $\lambda = -1$

$\Rightarrow W_1 = V_1 \oplus V_{-1}$

$$\left.\begin{aligned} H(x) &= H \cdot x = x \\ H(y) &= H \cdot y = -y \end{aligned}\right\} \Rightarrow V_1 \oplus V_{-1} \cong \mathbb{C} \cdot x \oplus \mathbb{C} \cdot y \cong \mathbb{C}^2$$

iii) W_2 : (W_2, ad)

$\rightarrow M(\mathrm{ad}_H) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \Rightarrow \lambda_1 = 0, \lambda_2 = 2, \lambda_3 = -2$

$\Rightarrow W_2 \cong V_2 \oplus V_0 \oplus V_{-2}$

A basis of $\mathrm{Sym}^2(\mathbb{C}^2)$ is given by $B = \{x^2, xy, y^2\}$

Then $ad_H(x^2) = x \cdot (H \cdot x) + (H \cdot x) \cdot x = 2x^2$

$ad_H(xy) = x(H \cdot y) + (H \cdot x) \cdot y = 0$

$ad_H(y^2) = y(H \cdot y) + (H \cdot y) \cdot y = -2y^2$

$\Rightarrow Sym^2(\mathbb{C}^2) \cong \mathbb{C} \cdot x^2 \oplus \mathbb{C} \cdot xy \oplus \mathbb{C} \cdot y^2 \cong V_2 \oplus V_0 \oplus V_{-2} = W_2$

In Fact: $W_m \cong Sym^m(\mathbb{C}^2)$