Nilpotent and solvable Lie algebras

Seminar on Lie algebras

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Agenda

Lie algebra

Let F be an arbitrary field throughout the discussion.

Definition

A **Lie algebra** L is a vector space L over a field F together with a binary operation

$$[\cdot, \cdot] : L \times L \to L,$$

called the corresponding (Lie-)**bracket** satisfying the following axioms:

$(L1)$ *The bracket operation is bilinear.*

$(L2)$ $[xx] = 0 \quad \forall x \in L.$

$(L3)$ $[x[yz]] + [y[zx]] + [z[xy]] = 0 \quad \forall x, y, z \in L.$

(L3) is often called the **Jacobi identity**.

Subalgebras & homomorphisms

Definition

A subspace K of L is called a (Lie-)**subalgebra** of L if

$$[xy] \in K \ \ \forall x, y \in K.$$

Definition

A linear map $\phi : L \to L'$ between two Lie algebras L and L' is called a **homomorphism** if

$$\phi[xy] = [\phi(x)\phi(y)] \quad \forall x, y \in L.$$

ϕ is a **isomorphism** $\Leftrightarrow$ ϕ bijective.

general linear algebra $\mathfrak{gl}(V)$

Example

Let V be a finite dimensional vector space over F. The space of all endomorphisms $End\ V$ is also a vector space.
Define a new operation $End\ V \times End\ V \to End\ V$ by

$$[x, y] = xy - yx$$

$$End\ V \longrightarrow \mathfrak{gl}(\mathbf{V})$$

$\mathfrak{gl}(\mathbf{V})$ is called **general linear algebra**.
Any subalgebra of $\mathfrak{gl}(V)$ is called a **linear Lie algebra**.
Fixing a basis for V one can identify $\mathfrak{gl}(V)$ with the set of all $n \times n$ matrices over F, denoted $\mathfrak{gl}(n, F)$.

linear Lie algebras

Example

$\mathfrak{sl}(V)$ – **special linear algebra** given by $\{x \in \mathfrak{gl}(V) \mid Tr(x) = 0\}$

$\mathfrak{t}(n, F)$ – set of all **upper triangular matrices** in $\mathfrak{gl}(n, F)$

$\mathfrak{n}(n, F)$ – set of all **strictly upper triangular matrices** in $\mathfrak{gl}(n, F)$

$\mathfrak{d}(n, F)$ – set of all **diagonal matrices** in $\mathfrak{gl}(n, F)$

Ideal, Center

Definition

An **ideal** I of a Lie algebra L is a subspace $I \subset L$ so that

$$x \in L,\ \ y \in I \Rightarrow [xy] \in I.$$

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Example

Trivial examples are 0 and L itself.
Another example for an ideal is the **center** $Z(L)$ of L defined by

$$Z(L) = \{z \in L \mid [xz] = 0 \ \ \forall x \in L\}.$$

Note that L is abelian precisely when its center is equal to L.

$\rightarrow$ definition of quotient algebra, homomorphism theorems

Normalizer

Definition

The **normalizer** of a subalgebra K of L is defined as:

$$N_L(K) = \{x \in L \mid [xK] \subset K\}$$

and is again a subalgebra of L.

Representations

Definition

Let V be a vector space over F. A **representation** of a Lie algebra L on V is a Lie algebra homomorphism

$$\phi : L \to \mathfrak{gl}(V)$$

Definition

The **adjoint representation** of a Lie algebra L is defined by $ad : L \to \mathfrak{gl}(L)$ which sends $x \to [x\ \cdot]$, so that

$$x.y = ad\ x\ (y) = [xy].$$

Lower central series

Definition

Let L be a Lie algebra. Consider a sequence of ideals of L defined by:

$$L^0 := L$$

$$L^{i+1} := [L\ L^i]$$

The sequence formed by the ideals of the form L^i is called **lower central series**.

So for example

$$L^1 = [L\ L]$$

and it is not difficult to verify that

$$[L^i\ L^j] \subset L^{i+j}.$$

Nilpotent Lie algebra

Definition

A Lie algebra L is called **nilpotent** if there exists an $n \in \mathbb{N}$, so that

$$L^n = 0$$

i.e. almost all terms of the lower central series vanish.

Proposition

Let L be a Lie algebra.

- **(i)** If L is nilpotent, then every subalgebra of L and all images of L under homomorphisms are nilpotent.
- **(ii)** If $L/Z(L)$ is nilpotent, then L is nilpotent.
- **(iii)** If L is nilpotent and nonzero, then $Z(L) \neq 0$.

ad-nilpotent element

Definition

Let L be a Lie algebra and $x \in L$. x is called **ad-nilpotent** if ad x is a nilpotent endomorphism.

Remark

When L is nilpotent, then for some $n \in \mathbb{N}$:

$$ad\ x_1\ ad\ x_2\ \dots\ ad\ x_n(y) = 0$$

for all $x_i, y \in L$. In particular, $(ad\ x)^n = 0$ for all $x \in L$.
Therefore, if L is nilpotent, then all elements of L are ad-nilpotent.

The following lemma will assist us later on.

Lemma

If $x \in \mathfrak{gl}(V)$ is a nilpotent endomorphism, then *ad* x is nilpotent as well.

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If $x \in \mathfrak{gl}(V)$ is a nilpotent endomorphism, then *ad x* is nilpotent as well.

To prove this lemma let

$$\lambda_x, \ \rho_x : End\ V \to End\ V$$

defined by $\lambda_x(y) = xy$ and $\rho_x(y) = yx$, called the left and right translation. λ_x and ρ_x commute are nilpotent, so $ad\ x = \lambda_x - \rho_x$ is nilpotent as well. □

Engel's theorem

Engel's theorem

If all elements of a Lie algebra L are ad-nilpotent, then L is nilpotent.

Helpful theorem

Theorem

Let $L \subset \mathfrak{gl}(V)$ be a subalgebra consisting of nilpotent endomorphisms and $V \neq 0$ finite dimensional. Then there exists a nonzero $v \in V$ with $L.v = 0$.

Engel's theorem

Engel's theorem

If all elements of a Lie algebra L are ad-nilpotent, then L is nilpotent.

Flag

Definition

Let V be a finite dimensional vector space of $dim\ V = n$. A **flag** in V is a chain of subspaces

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

where $dim V_i = i$. $x \in End\ V$ is said to **stabilise** this flag if for all i it holds true that:

$$x(V_i) \subset V_i.$$

Theorem

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Corollary

Let L be a subalgebra of $\mathfrak{gl}(V)$ and $V \neq 0$ finite dimensional.
If L consists of nilpotent endomorphisms, then there exists a flag (V_i) in V which is stabilised by L and which fulfils $L.V_i \subset V_{i-1}$ for all i.
In other words, the matrices of L, expressed in a suitable basis of V, all lie in $\mathfrak{n}(n, F)$, i.e. are all strictly upper triangular.

Derived series

Definition

Let L be a Lie algebra. Consider a sequence of ideals of L defined by:

$$L^{(0)} := L$$

$$L^{(i+1)} := [L^{(i)}\ L^{(i)}]$$

The sequence formed by the ideals of the form $L^{(i)}$ is called **derived series**.

Solvable Lie algebra

Definition

A Lie algebra L is called **solvable** if there exists an $n \in \mathbb{N}$ so that

$$L^{(n)} = 0$$

i.e. almost all terms of the derived series vanish.

Proposition

Let L be a Lie algebra.

- **(i)** Every subalgebra of L and all images of L under homomorphisms are solvable if L is solvable.
- **(ii)** If there exists a solvable ideal I of L so that L/I is solvable, then L is also solvable.
- **(iii)** If I, J are solvable ideals of L, then $I + J$ is solvable as well.

From now on assume that F is algebraically closed and has $char\ F = 0$.

Theorem

If $L \subset \mathfrak{gl}(V)$ is a solvable subalgebra and $V \neq 0$ is finite dimensional, then V contains a common eigenvector for all endomorphisms in L.

From now on assume that F is algebraically closed and has $char\ F = 0$.

Theorem

If $L \subset \mathfrak{gl}(V)$ is a solvable subalgebra and $V \neq 0$ is finite dimensional, then V contains a common eigenvector for all endomorphisms in L.

strategy of the proof:

- **(1)** find an ideal K of codimension one
- **(2)** verify that for K there exist common eigenvectors (by induction)
- **(3)** show that a space W of such eigenvectors is stabilised by L
- **(4)** for a $z \in L$ with $L = K + Fz$, find an eigenvector of z in W.

Lie's theorem

Lie's theorem

If $L \subset \mathfrak{gl}(V)$ is a solvable subalgebra and V is finite dimensional, then there exists a flag of V which is stabilised by L.
Put differently, there exists a basis V for which the corresponding matrices of L all lie in $\mathfrak{t}(n, F)$, i.e. are all upper triangular.

Corollaries

Corollary 1

If L is solvable, then there exists a chain of ideals

$$0 = L_0 \subset L_1 \subset \cdots \subset L_n = L$$

with $dim\ L_i = i$.

Corollary 2

If L is solvable, then $ad_L\ x$ is nilpotent for all $x \in [LL]$. Particularly, $[LL]$ is nilpotent.

References

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