Lie groups and Lie algebras

$$\mathbb{H} \supset S_3 \simeq \mathbf{SU}(2) \xrightarrow{2:1} \mathbf{SO}(3) \simeq \mathbb{R}P_3$$

$$\mathbf{SU}(2) = \left\{ A \in M(2;\mathbb{C}) \;\middle|\; \begin{array}{l} A^*A = I_2 \\ \det A = 1 \end{array} \right\} \qquad \mathbf{SO}(3) = \left\{ B \in M(3;\mathbb{R}) \;\middle|\; \begin{array}{l} B^tB = I_3 \\ \det B = 1 \end{array} \right\}$$

$$\begin{pmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos\theta - i\sin\theta \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2\theta) & -\sin(2\theta) \\ 0 & \sin(2\theta) & \cos(2\theta) \end{pmatrix}$$

$$\mathfrak{su}(2) \simeq \mathfrak{so}(3)$$

Seminar on Lie algebras
Heidelberg University
WiSe 2023

1. Matrix groups
2. The Lie algebra of a Lie group
3. Closed subgroups of a Lie group
4. The adjoint action and the adjoint representation

1. Matrix groups

$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

- Also called linear groups.
- These are subgroups of the general linear group

$$GL(n;\mathbb{K}) := \{ g \in M(n;\mathbb{K}) \mid \det g \neq 0 \}.$$

- More precisely, closed subgroups of $GL(n;\mathbb{K})$.

Examples

- $GL(n;\mathbb{C})$ itself.
- The special linear group
$$SO(n;\mathbb{K}) = \{ g \in GL(n;\mathbb{K}) \mid \det g = 1 \}.$$
- The orthogonal group
$$O(n;\mathbb{K}) = \{ g \in GL(n;\mathbb{K}) \mid g^t g = I_n \}.$$

Exercise

Show that these are indeed closed subgroups of $GL(n;\mathbb{K})$.

Groups defined by equations

- Further examples:

$$SO(n;\mathbb{K}),\quad Sp(2n;\mathbb{K}),\quad U(n),\quad \ldots$$

- In all these examples, the closed subgroup $G \subset GL(n;\mathbb{K})$ is defined by a system of <u>equations</u>:

$$G = \{ g \in GL(n;\mathbb{K}) \mid f(g) = c \}$$

where
$$\begin{aligned} f : GL(n;\mathbb{K}) &\longrightarrow \mathbb{K}^r \\ g &\longmapsto (f_1(g), \ldots, f_r(g)) \end{aligned}$$

- Examples:
  - $f(g) = \det g$ and $c = 1$ $(r = 1)$.
  - $f(g) = g^t g$ and $c = I_n$ $(r = n^2)$.

Tangent space at identity

$$G = \{ g \in GL(n; \mathbb{K}) \mid f(g) = c \}$$

$I_n \in G$, f differentiable

$$T_{I_n} G := \{ A \in M(n; \mathbb{K}) \mid f'(I_n) . A = 0 \}$$

$$= \ker f'(I_n) = \underbrace{\bigcap_{i=1}^{r} \ker f_i'(I_n)}_{\text{system of linear equations}}$$

Point: In all of our examples, the linear maps $f_1'(I_n), \dots, f_r'(I_n)$ are linearly independent.
[$f: GL(n; \mathbb{C}) \to \mathbb{K}^r$ is called a submersion]

Manifold structure

Assume that

$$G = \{ g \in GL(n; \mathbb{K}) \mid f(g) = c \}$$

with $f : GL(n; \mathbb{C}) \to \mathbb{K}^r$ a differentiable submersion and G a group.

Then G is a closed differentiable submanifold of $GL(n; \mathbb{K})$ and the differentiable maps

$$\mu : (g_1, g_2) \mapsto g_1 g_2 \quad \text{and} \quad \iota : g \mapsto g^{-1}$$

of $GL(n; \mathbb{K})$ induce differentiable maps

$$\mu : G \times G \to G \quad \text{and} \quad \iota : G \to G.$$

Lie groups

A **Lie group** is a group G endowed with a manifold structure with respect to which the group multiplication and inverse maps are morphisms.

Careful

Not all closed subgroups of $GL(n;\mathbb{C})$ are complex Lie groups: it depends whether they are defined by holomorphic equations. For instance, $U(n) := \{ g \in GL(n;\mathbb{C}) \mid \bar{g}^t g = I_n \}$ is a real Lie group but not a complex Lie group.

2. The Lie algebra of a Lie group

It turns out that, if G is a Lie group, the tangent space at 1_G has an induced Lie algebra structure, i.e. a bracket

$$[\cdot,\cdot] : \underline{g} \times \underline{g} \longrightarrow \underline{g}$$
$$(A, B) \longmapsto [A, B]$$

bilinear, anti-symmetric and satisfying the Jacobi identity:

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$$

For matrix groups

$$G \subset GL(n;\mathbb{K}) \underset{\text{abierto}}{\subset} M(n;\mathbb{K})$$

$$T_1 G \subset T_{I_n} GL(n;\mathbb{K}) = \underbrace{M(n;\mathbb{K})}$$

$[A,B] := AB - BA$ is a Lie bracket

$$G = \{ g \in GL(n;\mathbb{K}) \mid F(g) = c \}$$

$$\underline{\mathfrak{g}} := T_1 G = \{ A \in M(n;\mathbb{K}) \mid F'(I_n) \cdot A = 0 \}$$

stable under the above bracket ?

Examples

$$(\det)'(I_n)\cdot A$$

$$\underline{\mathfrak{sl}}(n;\mathbb{K}) = \{ A \in M(n;\mathbb{K}) \mid \underbrace{\operatorname{tr}}_{} A = 0 \}$$

$$\underline{\mathfrak{o}}(n;\mathbb{K}) = \{ A \in M(n;\mathbb{K}) \mid \underbrace{A^t + A}_{} = 0 \}$$

$$f(I_n + A) - f(I_n)$$

$$= (I_n + A)^t (I_n + A) - I_n^t I_n$$

$$= \underbrace{A^t + A}_{f'(I_n)\cdot A} + \underbrace{{}^tAA}_{= o(\|A\|^2)}$$

Exercise

Define $SO(n;\mathbb{K})$ and show that $\underline{\mathfrak{so}}(n;\mathbb{K}) = \underline{\mathfrak{o}}(n;\mathbb{K})$ but $SO(n;\mathbb{K}) \neq O(n;\mathbb{K})$.

For abstract Lie groups

$\rightsquigarrow$ why would $T_1 G$ have a Lie bracket?

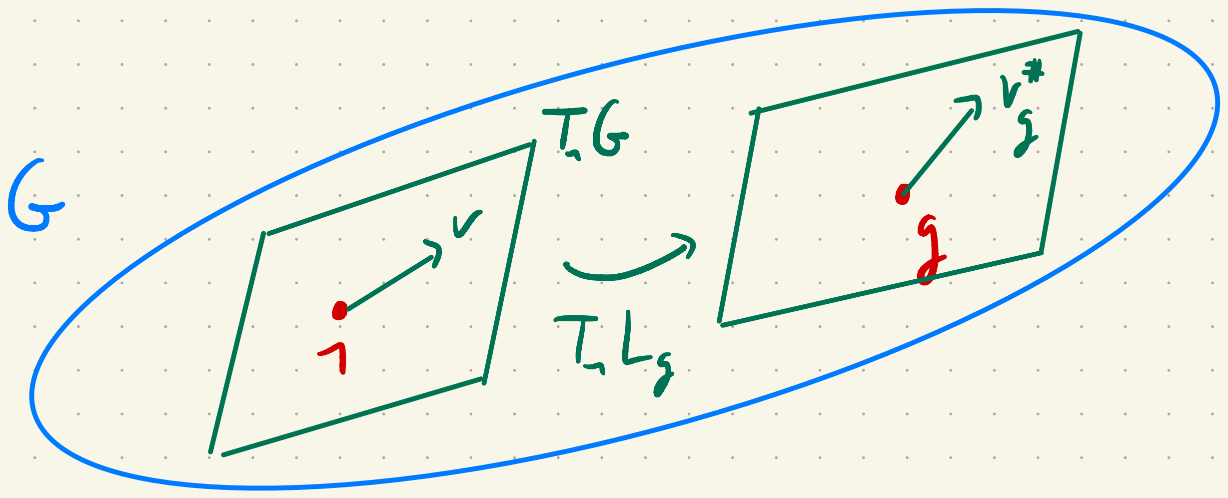
First, note that left and right translations define diffeomorphisms from G to itself.

Lemma Let G be a Lie group. Then to a tangent vector $v \in \underline{\mathfrak{g}} = T_1 G$, we can associate a vector field

$$v^\#_g := T_1 L_g \cdot v.$$

$\boxed{L_g(h) = gh}$

This vector field is left-invariant in the sense that $\forall\, h \in G$, $(L_h)_* v^\# = v^\#$.



A left-invariant vector field is entirely determined by its value at 1.

The map $v \longmapsto v^\#$ is linear and induces an isomorphism $T_1G \longrightarrow \mathfrak{X}_L(G)$ between the tangent space of G at 1 and the space of left-invariant vector fields on G. The inverse map sends

$$\underset{\mathfrak{X}_L(G)}{\underset{\Uparrow}{X}} \longmapsto \underset{T_1G}{\underset{\Uparrow}{X_1}}.$$

A proof that $v^\#$ is left-invariant

$$L_{g_1} \circ L_{g_2} = L_{g_1 g_2}$$
$$L_h^{-1} = L_{h^{-1}}$$

$$\left((L_h)_* v^\# \right)\Big|_g = T_{L_h^{-1}(g)} L_h \cdot v^\#_{L_h^{-1}(g)}$$

$$= T_{h^{-1}g} L_h \cdot v^\#_{h^{-1}g}$$

$$= \left(T_{h^{-1}g} L_h \circ T_1 L_{h^{-1}g} \right) \cdot v$$

by the chain rule!

$$= T_1 \left(L_h \circ L_{h^{-1}g} \right) \cdot v$$

$$= v^\#_g$$

Theorem The tangent bundle to a Lie group is trivial: the map

$$TG \longrightarrow G \times \underline{g}$$

$$(v \in T_g G) \longmapsto (g, T_g L_{g^{-1}} \cdot v)$$

is an isomorphism of vector bundles on G, whose inverse is

$$G \times \underline{g} \longrightarrow TG$$

$$(g, v) \longmapsto T_1 L_g \cdot v$$

The bracket of vector fields

① A vector field corresponds to a derivation on algebras of regular functions

$$X \in \mathfrak{X}(U), \quad f : U \to \mathbb{K}$$

$$\mathcal{L}_X f\,(x) := (df)_x\,(X_x) \in \mathbb{K}$$

the Lie derivative of f in the direction of X

$$\mathcal{L}_X(fg) = (\mathcal{L}_X f)\,g + f\,(\mathcal{L}_X g)$$

② The commutator of two derivations is a derivation and we define $[X, Y]$ via

$$\mathcal{L}_{[X,Y]} = \mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X$$

(3) The push-forward of vector fields by an automorphism satisfies

$$f_*[X,Y] = [f_* X, f_* Y]$$

so, in a Lie group, the commutator of two left-invariant vector fields is again left-invariant. This equips $\underline{g} = \mathfrak{X}_L(G)$ with a Lie algebra structure.

Exercise For $\underline{g} = M(n;\mathbb{K})$ and $A, B \in \underline{g}$, show that the above definition leads to $[A,B] = BA - AB$. Can you fix the sign?

3. Closed subgroups of a Lie group

Theorem (Elie Cartan)

Let H be closed subgroup of a real Lie group G. Then H is an embedded submanifold of G and a real Lie group itself.

(the case $G = GL(n;\mathbb{R})$ was proved earlier by John Von Neumann).

Let us focus on the case $G = GL(n; \mathbb{K})$.

Recall the existence of an exponential map

$$\exp : \mathfrak{gl}(n; \mathbb{K}) \longrightarrow GL(n; \mathbb{K})$$

$$A \longmapsto e^A := \sum_{n=0}^{+\infty} \frac{A^n}{n!}$$

The idea for the proof of the closed subgroup theorem is to define

$$\underline{\mathfrak{h}} := \{ X \in \underline{\mathfrak{g}} \mid \forall t \in \mathbb{R},\ \exp(tX) \in H \}$$

and use this to define local charts for H.

Consequences

Let G be a Lie group and let $\underline{g}$ be its Lie algebra.

- There is a bijective correspondence between

$$\left\{ \begin{array}{c} \text{sub-algebras of the} \\ \text{Lie algebra } \underline{g} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{connected subgroups} \\ \text{of } G \end{array} \right\}$$

- The Lie algebra g can also be seen as the space of <u>one-parameter subgroups of G</u>:

$$X = \frac{d}{dt}\Big|_{t=0} \exp(tX) \quad \text{in } \underline{g} = T_e G$$

and $\exp((s+t)X) = \exp(sX)\exp(tX)$ in G.

Note that a morphism of Lie groups

$$f : G_1 \longrightarrow G_2$$

(i.e. a differentiable group morphism)

induces a morphism of Lie algebras

$$df : \underline{\mathfrak{g}_1} \longrightarrow \underline{\mathfrak{g}_2}$$

such that the following diagram is commutative

$$\begin{array}{ccc} \underline{\mathfrak{g}_1} & \xrightarrow{df} & \mathfrak{g}_2 \\ \downarrow{\scriptstyle \exp_{G_1}} & & \downarrow{\scriptstyle \exp_{G_2}} \\ G_1 & \xrightarrow{f} & G_2 \end{array}$$

4. The adjoint action and the adjoint representation

A Lie group G acts on itself by conjugation

$$\left.\begin{array}{l} G \xrightarrow{\mathrm{Int}} \mathrm{Aut}_{gps}(G) \\ g \longmapsto \underbrace{\mathrm{Int}_g : (h \longmapsto ghg^{-1})}_{\text{group morphism}} \end{array}\right\}$$

Int is a group morphism !

$$\mathrm{Int}_{g_1 g_2} = \mathrm{Int}_{g_1} \circ \mathrm{Int}_{g_2}$$

$\forall g$, $\mathrm{Int}_g(1_G) = 1_G$, so the derivative of Int_g at 1_G induces bijective linear map $\underline{g} \longrightarrow \underline{g}$, denoted by Ad_g :

$$\mathrm{Ad}_g(X) = \left.\frac{d}{dt}\right|_{t=0} \mathrm{Int}_g(\exp(tX))$$

In matrix groups:

$$\mathrm{Ad}_g(X) = \frac{d}{dt}\Big|_{t=0} \left(g e^{tX} g^{-1}\right) = \underbrace{g X g^{-1}}_{\substack{\text{adjoint orbits} \\ = \text{ conjugacy classes}}}$$

Properties

$$\mathrm{Ad}_{g_1 g_2} = \mathrm{Ad}_{g_1} \circ \mathrm{Ad}_{g_2}$$

$$\mathrm{Ad}_g(\underbrace{[X, Y]}_{XY - YX}) = [\mathrm{Ad}_g X, \mathrm{Ad}_g Y]$$

Ad is a group morphism

$$\mathrm{Ad} : G \to \underbrace{\mathrm{Aut}_{\mathrm{Lie}}(\underline{g}) \subset \overbrace{GL(\underline{g})}^{\text{Lie group}}}_{\text{closed subgroup!}}$$

So $\mathrm{Aut}_{\mathrm{Lie}}(\underline{g})$ is a Lie group. As a consequence, the differential of Ad at 1_G is a morphism of Lie algebras

$$\mathrm{ad} := \mathrm{Ad}'(1_G) : \underline{g} \to \underbrace{\mathrm{Der}(\underline{g})}_{\substack{\text{The Lie algebra} \\ \text{of } \mathrm{Aut}_{\mathrm{Lie}}(\underline{g})}} \subset \overbrace{\mathrm{End}(\underline{g})}^{\substack{\text{Lie algebra} \\ \text{of } GL(\underline{g})}}$$

$$\mathrm{Der}(\underline{g}) := \left\{ u \in gl(\underline{g}) \;\middle|\; \begin{array}{l} \forall A, B \in gl(\underline{g}) \\ u(AB) = u(A)B + Au(B) \end{array} \right\}$$

Example $\quad \underline{g} \subset \underline{gl}(n; \mathbb{K}) \qquad \underbrace{u = [C, \cdot]}_{\text{called inner derivations}}$

$$[C, AB] = C(AB) - (AB)C$$

$$= \underbrace{(CA - AC)}_{[C,A]} B + A \underbrace{(CB - BC)}_{[C,B]}$$

In matrix groups:

$$\underbrace{\mathrm{ad}_Y X}_{= \mathrm{Ad}'_{1_G}(Y) \cdot X} = \frac{d}{dt}\Big|_{t=0} \exp(tY)\, X \exp(-tY)$$

$$= YX - XY = [Y, X]$$

Consequence: $\ker \mathrm{ad} = Z(\underline{g})$

The semisimple case

Facts For $\mathfrak{g}$ complex semisimple Lie algebra,

$\ker \mathrm{ad} = \{0\}$ and $\mathrm{Im}\, \mathrm{ad} = \mathrm{Der}(\mathfrak{g})$

Recall (holds in general)

$$\begin{tikzcd} \mathfrak{g}/\ker \mathrm{ad} \arrow[r, hook, "\mathrm{ad}"] \arrow[d, "\exp_G"'] & \mathrm{Der}(\mathfrak{g}) \arrow[d, "\exp_{\mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{g})}"] \\ G/\ker \mathrm{Ad} \arrow[r, hook, "\mathrm{Ad}"] & \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{g}) \end{tikzcd}$$

So, if G is connected and $\mathfrak{g}$ is semisimple,

$$G/\ker \mathrm{Ad} \cong \underbrace{\mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{g})_0}_{\mathrm{Int}(\mathfrak{g})}.$$

Example $PSL(n;\mathbb{C}) \cong \mathrm{Int}(\mathfrak{sl}(n;\mathbb{C}))$