REAL ALGEBRAIC VARIETIES

HEIDELBERG UNIVERSITY, WINTER SEMESTER 2022-2023

Target audience: Master students and advanced Bachelor students.
Instructor: Florent Schaffhauser.
Language of instruction: English.

Objectives

The purpose of this course is to introduce the notion of algebraic variety over a non-necessarily algebraically closed field, and study the basic properties of such objects. We will use a geometric approach and emphasise the most intuitively accessible examples of algebraic varieties, namely real algebraic curves and surfaces.

Concepts such as dimension and the tangent space will be introduced from the algebraic point of view and later related to their differential-geometric counterparts, in order to study the topological properties of real algebraic curves and surfaces, in their Hausdorff topology.

Along the way, we will also present a certain number of classical results from algebraic geometry, such as Bézout's theorem or the existence of 27 lines on a smooth cubic surface in 3-dimensional projective space.

Contents

As a complement to the material presented in the course, students are encouraged to follow the on *Seminar on sheaf theory* that will be run in Heidelberg during the Winter Semester 2022-2023.

Pre-requisites

- Polynomial rings and field extensions, notion of Galois group.
- Basic theory of commutative rings and their ideals.
- Concepts from point-set topology.

Towards the end of the course, we may use freely certain fundamental results on the classification of compact differentiable surfaces. However, these should not be regarded as a pre-requisite for the course, which is primarily intended as an introduction to the study of real algebraic varieties and the study of their Hausdorff topology.

No previous knowledge of algebraic geometry will be required for this course, and we will use only a minimal amount of commutative algebra. Instead, we will focus on geometric aspects of algebraic varieties. If you are interested in scheme theory and how schemes relate to varieties, you could benefit from following, simultaneously with this course, the *Algebraic Geometry 1* course taught by Dr Denis Vogel.

Evaluation

Depending on the number of students, the exam may be either written or oral (this will be announced within the first month of the course). To be admitted to the exam, you will need to collect at least 50% of the points from the graded problem sets (there will be a problem set every week, but it will be graded only every other week). The first question of the exam (whether written or oral) will be extracted from one of the non-graded problem sets, so you are encouraged not to neglect those.

Textbooks and supplementary reading

For complex varieties, we will mostly follow the standard texts [Hul03] (or the original German version [Hul00]) and [Per08], while for real varieties, we will draw from [BR90] and [BCR98], as well as, occasionally, from [Sha74].

If you want to follow up naturally on the geometric point of view adopted in this course, we recommend the books [Mum76] and [Mum99], while readers more interested in arithmetic aspects could probably benefit more directly from [Liu02].

References

[BCR98] Jacek Bochnak, Michel Coste, and Marie-Françoise Roy. *Real algebraic geometry*, volume 36 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1998. Translated from the 1987 French original, Revised by the authors.

[BR90] Riccardo Benedetti and Jean-Jacques Risler. *Real algebraic and semi-algebraic sets*. Actualités Mathématiques. [Current Mathematical Topics]. Hermann, Paris, 1990.

[Hul00] Klaus Hulek. *Elementare algebraische Geometrie*. Vieweg Studium: Aufbaukurs Mathematik. [Vieweg Studies: Mathematics Course]. Friedr. Vieweg & Sohn, Braunschweig, 2000. Grundlegende Begriffe und Techniken mit zahlreichen Beispielen und Anwendungen. [Basic concepts and techniques with various examples and applications].

[Hul03] Klaus Hulek. *Elementary algebraic geometry*, volume 20 of *Student Mathematical Library*. American Mathematical Society, Providence, RI, 2003. Translated from the 2000 German original by Helena Verrill.

[Liu02] Qing Liu. *Algebraic geometry and arithmetic curves*, volume 6 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné, Oxford Science Publications.

[Mum76] David Mumford. *Algebraic geometry. I*. Grundlehren der Mathematischen Wissenschaften, No. 221. Springer-Verlag, Berlin-New York, 1976. Complex projective varieties.

[Mum99] David Mumford. *The red book of varieties and schemes*, volume 1358 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, expanded edition, 1999. Includes the Michigan lectures (1974) on curves and their Jacobians, With contributions by Enrico Arbarello.

[Per08] Daniel Perrin. *Algebraic geometry*. Universitext. Springer-Verlag London, Ltd., London; EDP Sciences, Les Ulis, 2008. An introduction, Translated from the 1995 French original by Catriona Maclean.

[Sha74] I. R. Shafarevich. *Basic algebraic geometry*. Die Grundlehren der mathematischen Wissenschaften, Band 213. Springer-Verlag, New York-Heidelberg, 1974. Translated from the Russian by K. A. Hirsch.