Why adopt Type Theory as a Foundation of Mathematics from the perspective of the Minimalist Foundation

Maria Emilia Maietti

University of Padova

Formal proofs and synthetic mathematics,

Heidelberg University, 24-26/6/26

Abstract

- **Why, How, What about Type theory**

  as a foundation for constructive (!!!) maths

- some **peculiarities** distinguishing

Martin-Loef's Type Theory (base of **Agda**)
Homotopy Type Theory (base of **Cubical Agda**)
the Calculus of Inductive Constructions (base of **Rocq** and of **Lean**)

through the lens of the Minimalist Foundation (MF)

- **Open issues**

Why foundations of maths?

It is necessary

to choose a **language**

to formalize maths

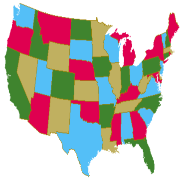
in a computer

Why foundations of maths?

It is necessary to choose a language

to formalize maths in a computer 

the language of TYPE THEORY had been chosen for

Four colour theorem 	in Rocq
the Feit-Thompson theorem of group theory	in Rocq
recent theorems by Fields Medalists (Vladimir Voevodsky, Peter Sholtze, Maryna Viazovska)	in Rocq and Lean

TYPE THEORY

INTERESTING FOR

computer scientists

mathematicians

why ?:

for **computer-aided formalization of proofs**

to produce **CORRECT PROOFS by construction!**

also thanks to **Autoformalization** with **AI-systems**

Why adopting TYPE THEORY ?

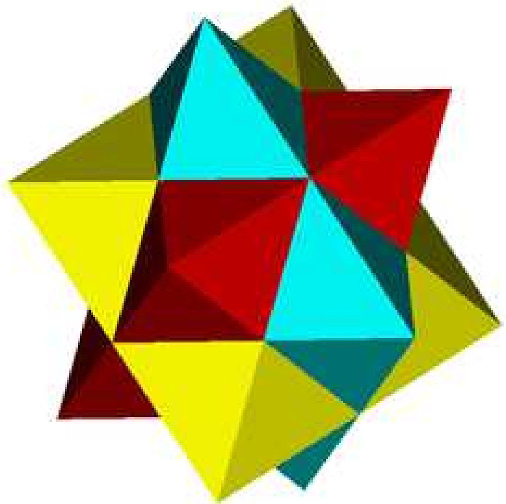
because

TYPE THEORY provides a **theoretical framework**

for the **foundation** of

mathematics	**logic**	**computer science**
set theory	**proof-theory**	**programming language**

Multi-identities of the concept of "type"



type theory	set theory	logic	programming language
$A\ type$	$A\ set$	$A\ prop$	A data-type
$a \in A$ term a of type A	element a of a set A	proof a of a proposition A	program a of specification A

Applications of Type Theory

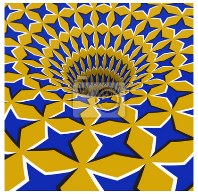
1. internal language of categories/models for synthetic mathematics

examples: Homotopy Type Theory, Synthetic computational mathematics in Rocq

2. Extraction of programs from proofs

   granted for constructive mathematics

examples: Martin-Löf's type theory, HoTT, Minimalist Foundation



3. different interpretation of logic as types

examples: *different notions* of existential quantifiers, hence choice principles

What do we mean by internal language of some categorical structures?

EXAMPLE **The internal language** of **elementary toposes**

is provided by a **generic calculus** TT_{topoi}

with an associated category $\mathrm{Th}(\mathsf{TT}_{\mathbf{topoi}})$ of **theories** and **translations** such that

1. Take the category ElTop of **elementary toposes** with **logical morphisms**

2. There exists a functor extracting the **internal theory** out of an elementary topos

$$\begin{array}{cccc} Int: & \mathrm{ElTop} & \to & \mathsf{Th}(\mathsf{TT}_{topoi}) \\ & \mathcal{E} & \mapsto & Int(\mathcal{E}) \end{array}$$

internal theory of $\mathcal{E}$

3. There exists a functor associating a syntactic elementary topos to a theory $\mathcal{T}$

$$\begin{array}{cccc} Syn: & \mathrm{Th}(\mathsf{TT}_{topoi}) & \to & \mathrm{ElTop} \\ & \mathcal{T} & \mapsto & Syn(\mathcal{T}) \end{array}$$

syntactic model of $\mathcal{T}$

for any elementary topos $\mathcal{E}$	for any theory T
$\mathcal{E} \simeq Syn(Int(\mathcal{E}))$	$\mathsf{T} \simeq Int(Syn(\mathsf{T}))$
equivalent categories	**equivalent theories**

Internal language *for* basic categorical structures

in

M.E. M. **Modular correspondence between dependent type theories and categories**, **2005**

internal languages 

as a **dependent type theory** **à la Martin-Löf** for

- **lex** categories
- **regular** categories
- **locally cartesian closed** categories
- **pretopoi**
- **arithmetic universes**
- **elementary topoi**

well known **Benabou-Mitchell language** for **elementary topoi**

does not provide ANY bi-equivalence

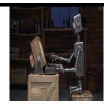
(because it is only a many simple typed logic)

But internal language of elementary toposes is for constructive maths

elementary toposes

as well as **Homotopy Type Theory** , **Martin-Löf's type theory**

and **dependent type theory CIC** of **Rocq**

are foundations for constructive mathematics 

because they validate ONLY **intuitionistic logic**

recalling that

classical logic = **intuitionistic logic** + **principle of Excluded Middle** $\phi \vee \neg\phi$

or + **law of Double-Negation** $\neg\neg\phi \rightarrow \phi$

WHILE

the dependent type theory of **Lean**

is a *a variant of* **the dependent type theory CIC** with **classical logic**

Essence of constructive mathematics

constructive mathematics

=

IMPLICIT computational mathematics

made EXPLICIT in computable models

validating *at least* Church thesis CT

for program-extraction from proofs

at least on Natural Numbers

A principle of computable mathematics: Church Thesis

$$(\mathbf{CT}) \qquad \frac{\forall x \in \mathsf{N}at \; \exists! \, y \in \mathsf{N}at \;\; R(x,y)}{\exists \mathbf{e} \in \mathsf{N}at \qquad (\, \forall x \in \mathsf{N}at \; \exists y \in \mathsf{N}at \;\; T(\mathbf{e},x,y) \;\; \& \;\; R(x, U(y)\,)\,)}$$

number-theoretic functional relations are all **computable**

(NOT valid in **classical maths**)

NOT VALID in **classical** maths

hence in **Lean** !!!

Constructive Mathematics by Bishop

Bishop 

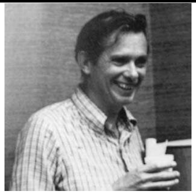
showed in "Foundations of constructive analysis"

that a large portion of functional analysis

can be reproduced constructively

$\Rightarrow$ in a computable way 

What foundation for constructive mathematics ??



Since the 80s various foundations for Bishop's constructive mathematics appeared including

Martin-Löf's type theory

Aczel set theory

Homotopy type theory

. . .

Why a minimalist foundation for constructive mathematics ??

j.w.w. Giovanni Sambin

We wanted to take advantage of the plurality

of **foundations** for **Bishop's constructive mathematics** 

Plurality of foundations $\Rightarrow$ need of a minimalist foundation

	classical	constructive
	ONE standard	NO standard
impredicative	Zermelo-Fraenkel set theory	internal theory of topoi Coquand-Huet's Calculus of Constructions
predicative	Feferman's explicit maths	Aczel's CZF Martin-Löf's type theory **HoTT** and Voevodsky's Univalent Foundations Feferman's constructive expl. maths

the **Minimalist Foundation** **MF**

A peculiarity of the Minimalist Foundation

from [Maietti'09] in agreement with [M. Sambin2005]

it is a TWO-LEVEL FOUNDATION

its intensional level mTT

(Minimalist Type Theory)

= a PREDICATIVE VERSION of Coquand-Huet's Calculus of Constructions CC

(fragment of Rocq)

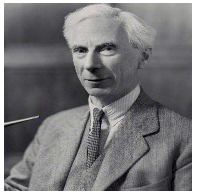
its extensional level emTT

(extensional Minimalist Type Theory)

is a PREDICATIVE QUASI-TOPOS

to avoid choice principles

Russell's notion of predicative definitions

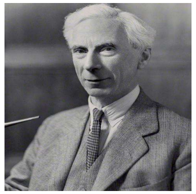


"Whatever involves an apparent variable

must *not be among the possible values* **of that variable."**

impredicativity of topos internal language

The generic internal language of elementary toposes

is impredicative in the sense of Russell  and Poincarè 

for the presence of the subobject classifier/ power-objects

because

for any formula $\phi(x, U)$

$$\{x \in Nat \mid \forall\, U \in \mathcal{P}(Nat) \;\; \phi(x, U)\,\} \,\varepsilon\, \mathcal{P}(Nat)$$

the power-object $\mathcal{P}(Nat)$ is closed under a subset

defined by a quantification over all subsets

including itself!

Need of a two level Foundation for constructive mathematics

a **Constructive Foundation** should

bridge

extensional set-theoretic language

(commonly used for abstract math)

proof-irrelevant

to communicate with people

intensional language

for **computer-aided proof checking**

proof-relevant

to communicate with machines

$\downarrow$

$\boxed{\text{realizability model for program extraction from proofs}}$

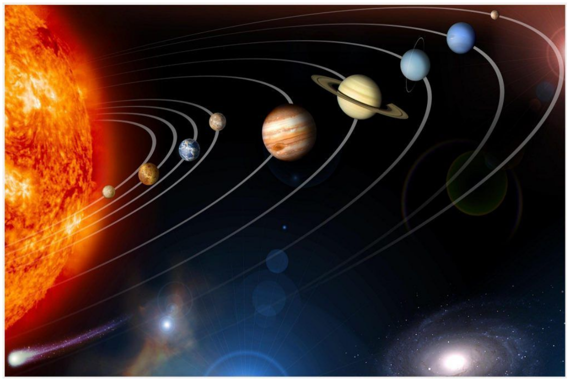
Notion of compatibility between theories

a theory T_1 is compatible with a theory T_2

iff

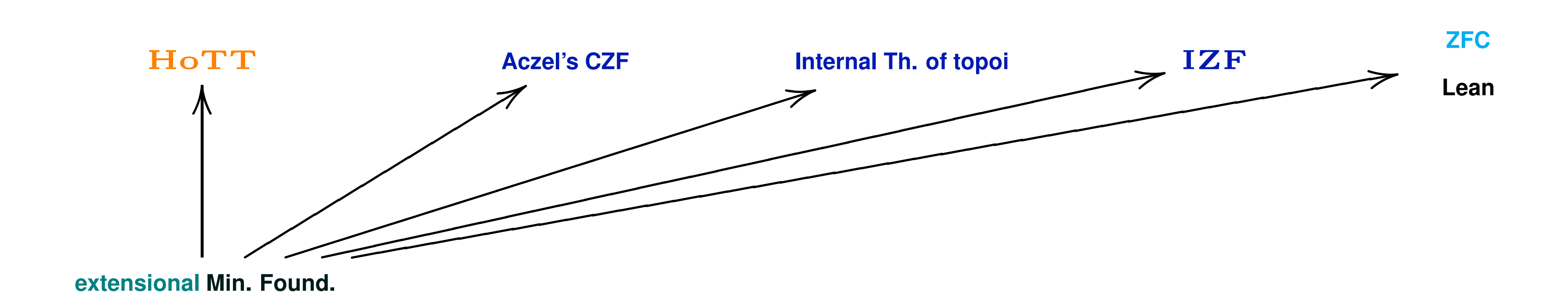
there is a translation $i\colon T_1 \longrightarrow T_2$

preserving the meaning of logical and set-theoretic operators

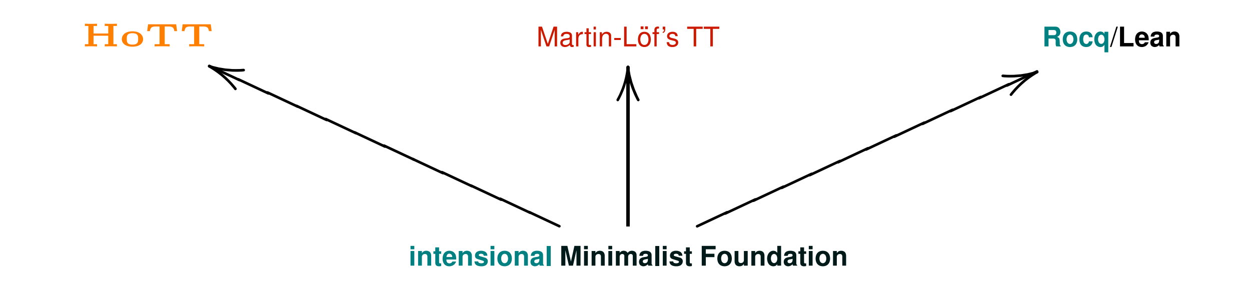


an advantage of the two-level structure: compatibility at the right level!

compatibility of our **extensional level** **eMF** with **extensional theories**



compatibility of with our **intensional level** **mtt** with intensional theories



Remarkable Peculiarity of HoTT

both levels of **MF** are **compatible** with **HoTT**

$$
\begin{tikzcd}
\boxed{\textbf{extensional level of MF}} \arrow[dr, "\text{MF-prop as h-prop}"] & \\
& \mathbf{HoTT} \\
\boxed{\textbf{intensional level of MF}} \arrow[ur, "\text{MF-prop as h-prop}"'] &
\end{tikzcd}
$$

M. Contente, M.E. Maietti. **The Compatibility of the Minimalist Foundation with Homotopy Type Theory.** **TCS 2024**

Remarkable Peculiarity of HoTT: it hosts the whole MF structure!

extensional level of MF

MF-prop as h-prop

HoTT

MF-prop as h-prop

intensional level of MF

MF-prop as set

extraction of programs

in **Martin-Loef's type theory**

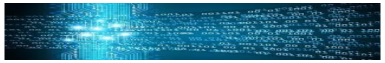
Differences between dependent type theories

Martin-Loef's Type Theory (base of **Agda**)
Homotopy Type Theory (base of **Cubical Agda**)
the Calculus of Constructions (base of **Rocq** and of **Lean**)

have **different ways**

to **identify propositions** with **types**

$\Rightarrow$ and hence **they differ on the validity some choice principles**

Within **Type theory** 

at least **three different conceptions of propositions:**

1. **Martin-Löf's Type Theory**

propositions as ALL **types**

2. the **HoTT**

propositions as **mono types**

i.e. types **with at most one proof**

3. the **Calculus of Constructions** /**MF**

propositions are **some types**

defined **primitively**

Within **Type Theory** 

at least **three different Existential quantifiers**

as a consequence of their conception of **propositions**

1. **Martin-Löf's Existential Quantifier**

identified with **the indexed sum type**

$$\Sigma x \in A\ \phi$$

2. the **h-propositional existential quantifier**

identified with the **truncated Martin-L"of's existential quantifier**

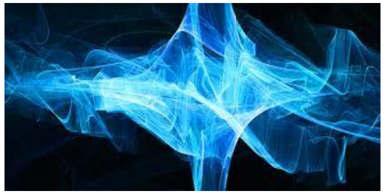
$\exists x \in A\ \phi = ||\Sigma x \in A\ \phi||$ as an **h-prop**

3. **primitive Intuitionistic existential quantifier**

$$\exists x \in A\ \phi$$

as in **CC**, hence Rocq, and **MF**

In TYPE THEORY two notions of function



a **PRIMITIVE** *notion* **of type-theoretic function**

$$f(x) \in B \; [x \in A]$$

$\neq$ **(syntactically)**

notion of functional relation

$$\forall x \in A \; \exists! y \in B \; R(x, y)$$

(NOT closed under "exponentiation")

In all mentioned theories

type-theoretic functions can be transformed into functional relations

taking their graph:

any type-theoretic function

$$f(x) \in B \; [x \in A]$$

(closed under ”exponentiation”)

$$\cap$$

notion of functional relation

$$\forall x \in A \; \exists! y \in B \; R(x, y)$$

taking the graph of $f(x) \in B \; [x \in A]$

Axiom of unique choice

$$\forall x \in A\, \exists! y \in B\, R(x,y) \quad \longrightarrow \quad \exists f \in A \to B\, \forall x \in A\, R(x, f(x))$$

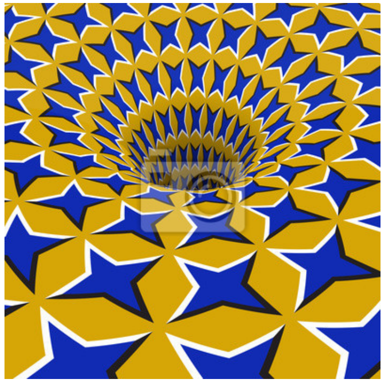
turns a functional relation into a type-theoretic function.

$\Rightarrow$ identifies the two distinct notions...

Axiom of choice

$$\forall x \in A\, \exists y \in B\, R(x, y) \quad \longrightarrow \quad \exists f \in A \to B\, \forall x \in A\, R(x, f(x))$$

a total relation contains the graph of a type-theoretic function.



Within **Dependent type theory** in general 

at least **THREE different Existential quantifiers:**

1. **Martin-Löf's Existential Quantifier**

identified with **the indexed sum type**

$$\Sigma x \in A\ \phi$$

$\Rightarrow$ **Axiom/Rule of choice holds**

2. the **h-propositional existential quantifier**

identified with **the truncated Martin-L"of's existential quantifier**

$\exists x \in A\ \phi = |\Sigma x \in A\ \phi|$ as an **h-prop**

$\Rightarrow$ **Axiom/Rule of UNIQUE choice holds**

because $\exists! x \in A\ \phi\ =\ \Sigma x \in A\ \phi$

3. **Intuitionistic existential quantifier** defined

as in **Rocq** or as in the **Minimalist Foundation**

$\Rightarrow$ **NO Axiom/Rule of UNIQUE choice holds**

Rule of choice

in a theory $\mathbf{T}$

if

$$\exists y \in B\ R(x, y)\ [x \in \Gamma]$$

is true in $\mathbf{T}$

$$\Downarrow$$

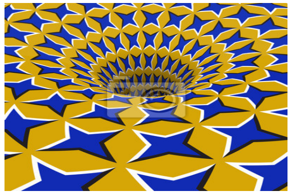
there exists a **choice** function term

$$f(x) \in B\ [x \in \Gamma]$$

in $\mathbf{T}$ such that

$$R(x, f(x))\ [x \in \Gamma]$$

is true in $\mathbf{T}$.



Rule of unique choice

in a theory $\mathbf{T}$

if

$$\exists! y \in B \; R(x,y) \; [x \in \Gamma]$$

is true in $\mathbf{T}$

$$\Downarrow$$

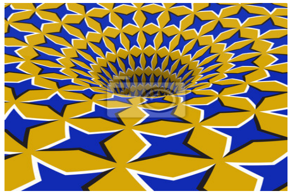
there exists a **choice** function term

$$f(x) \in B \; [x \in \Gamma]$$

in $\mathbf{T}$ such that

$$R(x, f(x)) \; [x \in \Gamma]$$

is true in $\mathbf{T}$.



Within **Dependent type theory** 

at least **THREE different Existential quantifiers**:

1. **Martin-Löf's Existential Quantifier**

identified with **the indexed sum type**

$$\Sigma x \in A\ \phi$$

$\Rightarrow$ **type-theoretic functions** $=$ **functional relations**

+ **Axiom of choice**

2. the **h-propositional existential quantifier**

identified with **the truncated Martin-L"of's existential quantifier**

$\exists x \in A\ \phi = |\Sigma x \in A\ \phi|$ as an **h-prop**

$\Rightarrow$ **type-theoretic functions** $=$ **functional relations**

(NO generic **Axiom of choice**)

3. **Intuitionistic existential quantifier** defined

as in **Rocq** or as in the **Minimalist Foundation**

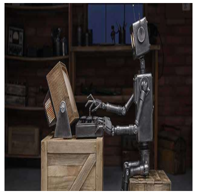
$\Rightarrow$ **type-theoretic functions** $\subseteq\neq$ **functional relations**

NO Axiom/Rule of UNIQUE choice holds

Peculiariy of classical MF/CC: base for synthetic computable mathematics

The **classical** versions of MF or **CC** (hence the corresponding fragment of **Lean**)

can **reconcile computable maths** with **classical maths**



$\Rightarrow$ base for **synthetic computable mathematics**:

Theorem (joint with P. Sabelli):

The classical versions of MF or CC

are **consistent** with **Theoretical Church Thesis**$(\mathbf{TCT})$

base for **synthetic computable mathematics**

Type-theoretic Church thesis

(**TCT**) $\forall f \in \mathsf{N}at{\rightarrow}\mathsf{N}at \quad \exists e \in \mathsf{N}at$

$$(\,\forall x \in \mathsf{N}at \;\; \exists y \in \mathsf{N}at \;\; T(e,x,y) \;\&\; U(y) =_{\mathsf{N}at} f(x)\,)$$

type-theoretic functions ($\subset_{\neq}$ **functional relations**)

are all **computable**

$\Rightarrow$ they can denote **computable** functions

as distinct from functional relations

Peculiariy of classical MF/CC: base for synthetic computable mathematics

Theorem (joint with P. Sabelli):

The fragment of Lean given by the classical MF/CC

are **consistent** with **Theoretical Church Thesis**$(\mathbf{TCT})$

base for **synthetic computable mathematics**: 

> **Proof.** the classical **MF**/**CC** can be interpreted
>
> in the quasi-topos of **Hyland**'s **assemblies**
>
> within **Hyland**'s **Effective Topos**
>
> thanks to

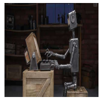
M. E. Maietti, P. Sabelli, Equiconsistency of the Minimalist Foundation with its classical version **APAL, 2025**

Peculiariy of classical MF/CC: base for synthetic computable mathematics

Theorem (joint with P. Sabelli):

The classical versions of MF or CC

are **consistent** with **Theoretical Church Thesis**$(\mathbf{TCT})$

base for **synthetic computable mathematics** 

While **HoTT** and **Martin-Löf's type theory**

are **CONTRADDITORY** with **TCT**

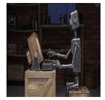
(as it implies **CT contraddittory** with **classical maths**)

since they do not distinguish

type-theoretic functions from **functional relations**

Peculiariy of MF/CC: base for synthetic computable mathematics + Brouwer's intuitionism

Both levels of MF and **CC** (hence the corresponding fragment of **Rocq**)

can **reconcile computable maths** with **Brouwer**'s **intuitionistic maths**

$\Rightarrow$ base for **synthetic intuitionistic computable mathematics a' la Brouwer:**

Theorem (joint with P. Sabelli):

The fragment of Lean given by the classical MF/CC

are **consistent** with **Theoretical Church Thesis**$(\mathbf{TCT})$

+ **Brouwer**'s **continuity** principles

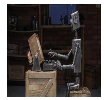
Peculiariy of MF/CC: base for synthetic computable mathematics + Brouwer's intuitionism

Theorem **(joint with P. Sabelli):**

Both levels of MF and CC

are **consistent** with **Theoretical Church Thesis**$(\mathbf{TCT})$

+ **Brouwer**'s **continuity** principles

$\Rightarrow$ base for **synthetic intuitionistic computable mathematics a' la Brouwer**  

Proof. **Both levels** of **MF** and **CC** can be interpreted

in the **quasi-topos** of **assemblies**

on a **constructive** metatheory + **Brouwer**'s **continuity** principles

thanks to

M. E. Maietti, P. Sabelli, and D. Trotta. Effectiveness and continuity in intuitionistic quasi-toposes of assemblies. MSCS, 2026

M. E. Maietti, P. Sabelli, Equiconsistency of the Minimalist Foundation with its classical version APAL, 2025

Brouwer's intuitionistic Mathematics

Brouwer's **intuitionistic continuity principles** include: 

Bar Induction (**BI**) =**spatiality** of **Baire locale**

(⇒ we can reason **inductively** **in the Baire space**)

where **Brouwer**'s **choice sequences**= **functional relations**

Local Continuity Principle (**LCP**)= **continuity of all functions** from **Baire space** to **Nat**

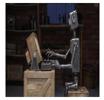
Brouwer's **LCP CONTRADICTS classical mathematics** 

Peculiariy of MF/CC: base for synthetic computable mathematics + Brouwer's intuitionism

Theorem (joint with P. Sabelli):

Both levels of **MF** and **CC**

are **consistent** with **Theoretical Church Thesis**$(\mathbf{TCT})$

$\Rightarrow$ base for **synthetic intuitionistic computable mathematics a' la Brouwer**  

While **HoTT** and **Martin-Löf's type theory**

are **CONTRADDITORY** with all the above principles

(as they do not distinguish **type-theoretic functions** from **functional relations**

Future work

- *Investigate the categorical counterpart of the translation*

of the two-level Minimalist Foundation in Hott

- *Generalize the* **tripos-to topos**/*quasi-topos constructions*

  predicatively

- **BIG goal**

Build **a proof-assistant for the Minimalist Foundation**