

Sheaves in HoTT

with applications to synthetic algebraic geometry

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Overview

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Ideal situation

We have T a theory¹ such that:

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Ideal situation

We have T a theory¹ such that:

1. T allows to study our spaces of interest synthetically.
2. T can be interpreted in some model.
3. The interpretation of synthetic results is as expected.

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Overview

We present the machinery used to build constructive models for synthetic algebraic geometry and synthetic Stone duality.

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- ▶ Our spaces of interest are build by gluing some basic spaces².
- ▶ These basic spaces have an algebra/geometry duality.

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We present the machinery used to build constructive models for synthetic algebraic geometry and synthetic Stone duality.

What makes it work?

- ▶ Our spaces of interest are build by gluing some basic spaces².
- ▶ These basic spaces have an algebra/geometry duality.

What prerequisite to use it?

- ▶ You don't need technical ∞ -categories know-how.
- ▶ You do need HoTT and its lex modalities.

²More precisely, they are sheaves on the category of basic spaces.

Introduction to synthetic geometry

Presheaf models

Sheaf models

Applications to synthetic algebraic geometry

Perspective

Synthetic geometry (CoLD version)

We want to study some spaces synthetically
(e.g. smooth manifolds, sequential spaces, schemes).

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Remark

Historical approaches didn't use MLTT.

What do we get?

Slogan

- ▶ All types come with a topology/smooth structure/etc.
- ▶ All maps are continuous/smooth/etc.

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Remark

This means we have to abandon LEM and AC.

Examples of synthetic geometries

Name	Spaces of interest
Synthetic differential geometry ³	Smooth manifolds
Synthetic domain theory ⁴	Domains
Synthetic topology ⁵	Sequential spaces
Synthetic algebraic geometry ⁶	Schemes
Condensed type theory ⁷ / Synthetic Stone duality ⁸	Stone spaces, locally compact Hausdorff spaces

³[Lawvere 67], [Kock 77], [Dubuc 79], etc.

⁴[Hyland 91], [Fiore, Plotkin 96], etc.

⁵[Escardó 04], [Lešnik 10], etc.

⁶[Bleschmidt 17], [Cherubini, Coquand, Hutzler 24], etc.

⁷[Barton, Commelin 24]

⁸[Cherubini, Coquand, Geerlings, Moeneclaey 24]

Synthetic homotopy theory

We want to study ∞ -groupoids (aka homotopy types) synthetically.

Method

1. MLTT can be interpreted in ∞ -groupoids.
2. This validates HoTT (= MLTT + univalence + HITs).
3. Work in HoTT.

Synthetic Geometry (HoTT version)

Method

1. Embed the spaces of interest in an ∞ -topos.
2. Interpret HoTT in this ∞ -topos [Shulman 19].
3. Find **new axioms** validated by this interpretation.
4. Work in HoTT + **these axioms**.

The spaces of interest are then **sets** satisfying some proposition.

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The spaces of interest are then **sets** satisfying some proposition.

Why do this?

- ▶ To use some homotopy theory (e.g. shape or cohomology).
- ▶ For the convenience of univalence and HITs.

Examples

Name	Spaces of interest
Real-cohesive homotopy type theory ⁹	Locally contractible spaces and their homotopy types
Synthetic algebraic geometry ¹⁰	Schemes
Condensed type theory ¹¹ / Synthetic Stone duality ¹²	Stone spaces, locally compact Hausdorff spaces
Simplicial type theory ¹³ / Triangulated type theory ¹⁴	Higher categories

⁹[Shulman 18]

¹⁰[Bleschmidt 17], [Cherubini, Coquand, Hutzler 24], etc.

¹¹[Barton, Commelin 24]

¹²[Cherubini, Coquand, Geerlings, Moeneclaey 24]

¹³[Riehl, Shulman 17], etc.

¹⁴[Gratzer, Weinberger, Buchholtz 24]

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How to find the new axioms?

Need axioms that are:

- ▶ Validated by the interpretation of HoTT into this topos.
- ▶ Enough to define and study our spaces of interest.

This is usually very challenging.

Algebra/geometry dualities

Name	Algebra	Geometry
Zariski duality	Rings	Affine schemes
Stone duality	Boolean algebras	Stone spaces
//	Distributive lattices	Coherent spaces
Gelfand duality	C^* -algebras	Compact Hausdorff spaces
⋮	⋮	⋮

Back to our goal

Let's focus on synthetic algebraic geometry.

Goal

We want to work synthetically with schemes.

Basic spaces are affine schemes.

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Core idea

Leverage the rings/affine schemes duality to get nice axioms validated by the relevant topoi.

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We want to work synthetically with schemes.

Basic spaces are affine schemes.

Core idea

Leverage the rings/affine schemes duality to get nice axioms validated by the relevant topoi.

- ▶ [Kock 80] for external duality used in 1-topoi.
- ▶ [Bleschmidt 17] for internal duality in 1-topoi.
- ▶ [Coquand, Höfer, Sattler 26] for internal duality in ∞ -topoi.

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Perspective

This section presents results from [Coquand, Höfer, Sattler 26].

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We assume given $\mathbb{T}$ an algebraic theory.

Idea

$\mathbb{T}$ models the duals to our basic spaces.

For synthetic algebraic geometry, $\mathbb{T}$ is the theory of rings.

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$\mathbb{T}$ models the duals to our basic spaces.

For synthetic algebraic geometry, $\mathbb{T}$ is the theory of rings.

Intermediary goal

Work synthetically in the ∞ -topos of presheaves on $(\mathbb{T}\text{-Alg}_{fp})^{op}$.

Need axioms satisfied by the interpretation of HoTT in this topos.

The generic object

We have a presheaf sending any $A : \mathbb{T}\text{-Alg}_{fp}$ to its underlying set. It is a $\mathbb{T}$ -algebra in $\text{PSh}((\mathbb{T}\text{-Alg}_{fp})^{op})$.

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Axiom 1 (Generic object)

There is a set $\mathbb{G}$ with a $\mathbb{T}$ -algebra structure.

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Axiom 1 (Generic object)

There is a set $\mathbb{G}$ with a $\mathbb{T}$ -algebra structure.

For synthetic algebraic geometry, we have a ring R .

The spectrum construction

Definition

A $\mathbb{G}$ -algebra is a $\mathbb{T}$ -algebra A with a morphism from $\mathbb{G}$ to A .

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Example from synthetic algebraic geometry

Given $P : R[X]$, then

$$\text{Spec}(R[X]/P) = \text{Hom}_{R\text{-Alg}}(R[X]/P, R) = \{x : R \mid P(x) = 0\}.$$

Duality axiom

Axiom 2 (Duality)

Given $A : \mathbb{G}\text{-Alg}_{fp}$, the map

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The map

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Definition

Sets in the image of Spec are called **affines**.

Corollary

Given $A, B : \mathbb{G}\text{-Alg}_{fp}$, we have that

$$(\text{Spec}(B) \rightarrow \text{Spec}(A)) = \text{Hom}_{\mathbb{G}\text{-Alg}}(A, B).$$

Corollary

Given $A, B : \mathbb{G}\text{-Alg}_{fp}$, we have that

$$(\text{Spec}(B) \rightarrow \text{Spec}(A)) = \text{Hom}_{\mathbb{G}\text{-Alg}}(A, B).$$

Example from synthetic algebraic geometry

$$(R \rightarrow R) = \text{Hom}_{R\text{-Alg}}(R[X], R[X]) = R[X]$$

We know that representables in a presheaf category are projective.

Choice for affines

We know that representables in a presheaf category are projective.

Axiom 3 (Choice for affines)

Given $A : \mathbb{G}\text{-Alg}_{fp}$, the set $\text{Spec}(A)$ has choice.

Theorem [Coquand, Höfer, Sattler 26]

Given $\mathbb{T}$ an algebraic theory, there exists a model of HoTT where:

- ▶ There is a $\mathbb{T}$ -algebra $\mathbb{G}$.
- ▶ For all $A : \mathbb{G}\text{-Alg}_{fp}$, the map $A \rightarrow \mathbb{G}^{\text{Spec}(A)}$ is an equivalence.
- ▶ For all $A : \mathbb{G}\text{-Alg}_{fp}$, the set $\text{Spec}(A)$ has choice.

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Toward sheaves

We want to work with sheaves instead of presheaves.

External approach

Redo [Coquand, Höfer, Sattler 26] with sheaves.

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Forget about higher category theory, we can do it in HoTT!

Internal approach

We define sheaves in $(\text{HoTT} + \text{presheaf axioms})$.

Internal topologies

Definition

A **topology** consists of a class $\mathbb{J}$ of finite sums of affines such that $1 \in \mathbb{J}$ and $\mathbb{J}$ is stable under Σ .

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A **$\mathbb{J}$ -cover** is a map with fibers in $\mathbb{J}$.

Given S affine, any $\mathbb{J}$ -cover over S is merely of the form

$$U_1 + \cdots + U_n \rightarrow S$$

with each U_i affine.

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with each U_i affine.

Lemma

- ▶ Isomorphisms are $\mathbb{J}$ -cover.
- ▶ $\mathbb{J}$ -covers are stable under pullbacks and composition.

Internal sheaves

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Given a $\mathbb{J}$ -cover $T \rightarrow S$ and a set X that is a $\mathbb{J}$ -sheaf, the map

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Remark

There is a version for n -types based on iterated joins [Williams 25].

Internal sheaves

Definition

A property of types Q is an **accessible lex modality** if there exists a small family of propositions $(P_i)_{i:I}$ such that

$$Q(X) \Leftrightarrow \text{For all } i : I \text{ the map } X \rightarrow X^{P_i} \text{ is an equivalence.}$$

So being a $\mathbb{J}$ -sheaf is an accessible lex modality.

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Corollary [Rijke, Shulman, Spitters 20]

- ▶ $\mathbb{J}$ -sheaves are stable under Σ and identity types.
- ▶ $\mathbb{J}$ -sheaves are stable under Π_X for any type X .
- ▶ We have a $\mathbb{J}$ -sheafification.
- ▶ The type of $\mathbb{J}$ -sheaves is itself a $\mathbb{J}$ -sheaf.

Lex modalities and submodels of HoTT

Lex modalities correspond to ∞ -subtopoi.

So $\mathbb{J}$ -sheaves should form a model of HoTT.

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Theorem [Quirin 16]

Given an accessible lex modality L , we have an interpretation $\llbracket _ \rrbracket$ of HoTT into L -modal types where:

$$\begin{aligned}\llbracket \text{Type} \rrbracket &= \text{Type}_L \\ \llbracket \Sigma_X Y \rrbracket &= \Sigma_{\llbracket X \rrbracket} \llbracket Y \rrbracket \\ \llbracket \Pi_X Y \rrbracket &= \Pi_{\llbracket X \rrbracket} \llbracket Y \rrbracket \\ \llbracket x =_X y \rrbracket &= \llbracket x \rrbracket =_{\llbracket X \rrbracket} \llbracket y \rrbracket \\ \llbracket \lVert X \rVert \rrbracket &= L(\lVert \llbracket X \rrbracket \rVert)\end{aligned}$$

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So we have a $\mathbb{J}$ -sheaf model of HoTT.

The sheaf axioms: generic object

What does the $\mathbb{J}$ -sheaf model preserve from the presheaf axioms?

The sheaf axioms: generic object

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Definition

A topology $\mathbb{J}$ is **subcanonical** if $\mathbb{G}$ is a $\mathbb{J}$ -sheaf.

From now on we assume $\mathbb{J}$ subcanonical.

The sheaf axioms: generic object

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Lemma (Generic object in sheaves)

In the $\mathbb{J}$ -sheaf model, there is a $\mathbb{T}$ -algebra $\mathbb{G}$.

The sheaf axioms: generic object

So we have $\mathbb{G}$ in the $\mathbb{J}$ -sheaf model. Now we want to have $\mathbb{J}$.

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Definition

A topology is **simply defined** if:

- ▶ $\mathbb{J}$ can be defined from $(\text{HoTT} + \text{there is a } \mathbb{T}\text{-algebra } \mathbb{G})$.
- ▶ $\llbracket S \in \mathbb{J} \rrbracket_{\mathbb{J}}$ implies $L_{\mathbb{J}}(S \in \mathbb{J})$.

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Lemma

In the $\mathbb{J}$ -sheaf model, for all $S \in \mathbb{J}$ we have $\llbracket S \rrbracket$.

The sheaf axioms: duality

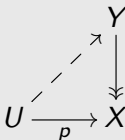
Lemma (Duality in sheaves)

In the $\mathbb{J}$ -sheaf model, duality holds.

The sheaf axioms: local choice

Definition

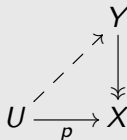
A type X has **$\mathbb{J}$ -local choice** if given a surjection $Y \twoheadrightarrow X$, there exists a $\mathbb{J}$ -cover $p : U \rightarrow X$ with a dotted arrow



The sheaf axioms: local choice

Definition

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Lemma (Local choice)

In the $\mathbb{J}$ -sheaf model, affines have $\mathbb{J}$ -local choice.

Theorem

Assume given:

- ▶ $\mathbb{T}$ an algebraic theory
- ▶ $\mathbb{J}$ a subcanonical and simply defined topology.

Then there exists a $\mathbb{J}$ -sheaf model of HoTT where:

- ▶ There is a $\mathbb{T}$ -algebra $\mathbb{G}$ such that for all $S \in \mathbb{J}$ we have $\|S\|$.
- ▶ For all $A : \mathbb{G}\text{-Alg}_{fp}$, the map $A \rightarrow \mathbb{G}^{\text{Spec}(A)}$ is an equivalence.
- ▶ For all $A : \mathbb{G}\text{-Alg}_{fp}$, the set $\text{Spec}(A)$ has $\mathbb{J}$ -local choice.

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The presheaf model

Proposition

We have a model of HoTT where:

- ▶ There is a ring R .
- ▶ For all $A : R\text{-Alg}_{fp}$ the map $A \rightarrow R^{\text{Spec}(A)}$ is an equivalence.
- ▶ For all $A : R\text{-Alg}_{fp}$ the set $\text{Spec}(A)$ has choice.

Definition

The Zariski topology consists of the types merely of the form

$$x_1 \text{ inv} + \cdots + x_n \text{ inv}$$

where $(x_1, \dots, x_n) = 1$ in R .

Remark

R is local if and only if for all $S \in \text{Zar}$ we have $\|S\|$.

The Zariski sheaf model

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- ▶ For all $A : R\text{-Alg}_{fp}$ the set $\text{Spec}(A)$ has Zariski-local choice.

An interlude: sheafifying in a sheaf model

Remark

We considered a topology internal to presheaves.

The same story works for a topology internal to sheaves.

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Remark

In Zariski sheaves, affines are stable under finite sums.

Now we work in Zariski sheaves and replace finite sums of affines by affines in the definition of topology.

Definition

A type X is formally étale if for all $\epsilon : R$ nilpotent, the map

$$X \rightarrow X^{\epsilon=0}$$

is an equivalence.

Étale topology

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The étale topology consists of the non-empty formally étale affines.

Proposition

A type X is an étale sheaf if and only if the map

$$X \rightarrow X^{\|\mathrm{Spec}(R[X]/P)\|}$$

is an equivalence for all P monic unramifiable.

The étale sheaf model

Proposition

We have a model of HoTT where:

- ▶ There is a local ring R such that any monic unramifiable polynomial in R has a root.
- ▶ For all $A : R\text{-Alg}_{fp}$ the map $A \rightarrow R^{\text{Spec}(A)}$ is an equivalence.
- ▶ For all $A : R\text{-Alg}_{fp}$ the set $\text{Spec}(A)$ has étale-local choice.

The fppf topology

Definition

An affine $\mathrm{Spec}(A)$ is flat if the functor $A \otimes_R -$ is exact.

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The fppf topology consists of the non-empty flat affines.

Remark

Given P monic non-constant, $\mathrm{Spec}(R[X]/P)$ is non-empty and flat. So in fppf sheaves, monic non-constant polynomials have roots.

Remark

Formally étale affines are flat, so fppf sheaves are étale sheaves.

The fppf sheaf model

Proposition

We have a model of HoTT where:

- ▶ There is a local ring R such that any monic non-constant polynomial in R has a root.
- ▶ For all $A : R\text{-Alg}_{fp}$ the map $A \rightarrow R^{\text{Spec}(A)}$ is an equivalence.
- ▶ For all $A : R\text{-Alg}_{fp}$ the set $\text{Spec}(A)$ has flat-local choice.

The four models compared

Topology	The ring R is	Affines have
-	arbitrary	full choice
Zariski	local	Zariski-local choice
Étale	local + monic unramifiable have roots	étale-local choice
Fppf	local + monic non-constant have roots	flat-local choice

Outline

Introduction to synthetic geometry

Presheaf models

Sheaf models

Applications to synthetic algebraic geometry

Perspective

Synthetic Stone duality

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Simplifications

- ▶ 2 is the only local boolean ring.
So in Zariski sheaves $\mathbb{G} = 2$.
- ▶ Any boolean ring is formally étale and flat.
So fppf topology = étale topology = $\{S \mid \neg\neg S\}$.

A model for synthetic Stone duality

Theorem

We have a model of HoTT where:

- ▶ For all $A : \text{Bool}_{cp}$, if $\neg\neg\text{Spec}(A)$ then $\|\text{Spec}(A)\|$.
- ▶ For all $A : \text{Bool}_{cp}$, the map $A \rightarrow 2^{\text{Spec}(A)}$ is an equivalence.
- ▶ For all $A : \text{Bool}_{cp}$, the set $\text{Spec}(A)$ has local choice.

Triangulated type theory¹⁵

Let $\mathbb{T}$ be the theory of distributive lattice.

Let $\mathbb{L}$ be the generic such lattice.

Definition

Consider the topology generated by:

- ▶ If $0 =_{\mathbb{L}} 1$ we add $\perp$.
- ▶ For all $i, j : \mathbb{L}$ we add $i \leq j + j \leq i$.

¹⁵[Gratzer, Weinberger, Buchholtz 24]

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We should get a model of HoTT with:

- ▶ $\mathbb{L}$ a totally ordered bounded set.
- ▶ Duality for finitely presented distributive lattice over $\mathbb{L}$.
- ▶ Some form of local choice for affines.

¹⁵[Gratzer, Weinberger, Buchholtz 24]

Summary

Given **spaces of interests** build by **gluing basic spaces** such that:

- ▶ These basic spaces enjoy a **duality** with $\mathbb{T}$ -algebras.
- ▶ The topology $\mathbb{J}$ encoding the gluing can be defined internally.
- ▶ $\mathbb{J}$ is subcanonical and simply defined.

Then we have a model of HoTT where:

- ▶ We have a $\mathbb{T}$ -algebra $\mathbb{G}$.
- ▶ $\mathbb{J}$ -covers are surjective.
- ▶ We have a **duality** between f.p. $\mathbb{G}$ -algebras and affines.
- ▶ Affines have $\mathbb{J}$ -local choice.

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In SAG and SSD, these axioms are enough for substantial results.