

Tutorial 7 – Ternary semantics

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Let us consider the natural deduction rules for well-formed formulas constructed in the lectures. Here, we will consider the following constructors for well-formed formulas.

- $P : \text{atoms} \rightarrow \text{Wff}$
- $\wedge : \text{Wff} \rightarrow \text{Wff} \rightarrow \text{Wff}$
- $\vee : \text{Wff} \rightarrow \text{Wff} \rightarrow \text{Wff}$

In this problem set, we want to define new semantics for well-formed formulas and study the soundness of that interpretation.

QUESTION 1. Consider a set S with three elements. We can call them `false`, `undefined` and `true` (or `-1`, `0`, `1`). Using Rocq syntax, give an inductive definition of this set. You can use the prover to type-check your answer, but do write the code in pen-and-paper form in your solution.

QUESTION 2. (Interpreting conjunction) Using Rocq syntax, define a binary operator `andS` : $S \rightarrow S \rightarrow S$ according to the following table.

s	s'	andS s s'
false	false	false
false	undefined	false
false	true	false
undefined	false	false
undefined	undefined	undefined
undefined	true	undefined
true	false	false
true	undefined	undefined
true	true	true

QUESTION 3. (Interpreting disjunction) Noting the analogy between `andS` and the `min` function on $\{-1, 0, 1\}$, define a function `orS` : $S \rightarrow S \rightarrow S$ (in analogy with `max`) that will later be used to interpret disjunction. Write a table for it and give the corresponding Rocq code that defines it.

QUESTION 4. (Soundness) Let us keep the same definition of entailment as in Boolean semantics, using `true` as a marker for semantic validity. Does the above interpretation of $\wedge$ and $\vee$ provide sound semantics for natural deduction?

QUESTION 5. (No-contradiction) Think of $\text{negS} : S \rightarrow S$ as the function that sends $t \in \{-1, 0, 1\}$ to $-t$ (with the usual convention that $-0 = 0$ and $-(-1) = 1$). Show that, there exists a t in S such that $\text{negS} (\text{andS } (t (\text{negS } t))) \neq \text{true}$.

QUESTION 6. (Falsity and negation) Discuss adding a constructor $\perp : \text{Wff}$ and an operation $\neg : \text{Wff} \rightarrow \text{Wff}$ for well-formed formulas. What happens when we do?