Tutorial 1 – Well-formed formulas

Due by May 11th at 4 PM.

Exercise 1. (Well-formed formulas) Are the following expressions well-formed formulas? Pick the ones that are, give their tree representation and compute their height and list of strict sub-formulas. For the formulas which are not well-formed, can you suggest a rule that would make them well-formed?

a. $\neg\neg P_0$

b. $P_2 \Leftarrow P_1$

c. $(P_1 \Rightarrow P_2) \Rightarrow P_3$

d. $P_1 \Rightarrow P_2 \Rightarrow P_3$

e. $(P_1 \vee P_2) \wedge \neg P_3$

f. $P_1 \vee P_2 \wedge \neg P_3$

g. $\neg P_1 \wedge P_2 \Rightarrow \neg(P_1 \wedge P_2)$

h. $P_1 \neg \wedge P_2$

Exercise 2. Let $\mathscr{B}$ be a set of basic propositions. Let $\mathscr{V} := \mathscr{B} \cup \{\Rightarrow, \wedge, \vee, \Leftrightarrow, \neg\} \cup \{(\} \cup \{)\}$ be an alphabet and let $\mathscr{V}^*$ be the set of words on the alphabet $\mathscr{V}$. In Lecture 1, we defined the set $\mathscr{F}(\mathscr{V})$ of *propositional formulas* based on the alphabet $\mathscr{V}$ is the smallest subset $\mathscr{W} \subset \mathscr{V}^*$ satisfying the following properties:

1. $\mathscr{B} \subset \mathscr{W}$.
2. If $F \in \mathscr{W}$ and $F' \in \mathscr{W}$, then, for all $\diamondsuit \in \{\Rightarrow, \wedge, \vee, \Leftrightarrow\}$, $(F \diamondsuit F') \in \mathscr{W}$.
3. If $F \in \mathscr{W}$ then $(\neg F) \in \mathscr{W}$.

Explicitly, we let $\mathscr{E}$ be the set of all such subsets $\mathscr{W}$ of $\mathscr{V}^*$ (meaning all subsets $\mathscr{W} \subset \mathscr{V}^*$ satisfying properties a, b and c above) and we set:

$$\mathscr{F}(\mathscr{V}) := \bigcap_{\mathscr{W} \in \mathscr{E}} \mathscr{W}.$$

Then in Lecture 2, we defined a subset $\mathscr{F} \subset \mathscr{V}^*$ by constructing a sequence $(\mathscr{F}_n)_{n\in\mathbb{N}}$ of subsets of $\mathscr{V}^*$ that was defined as follows

1. $\mathscr{F}_0 := \mathscr{B}$.
2. $\mathscr{F}_{n+1} :=$

$$\mathscr{F}_n \cup \left\{\neg F \text{ where } F \in \mathscr{F}_n\right\} \cup \left\{F \diamondsuit F' \text{ where } F, F' \in \mathscr{F}_n \text{ and } \diamondsuit \in \{\Rightarrow, \wedge, \vee, \Leftrightarrow\}\right\}.$$

and then setting

$$\mathscr{F} := \bigcup_{n\in\mathbb{N}} \mathscr{F}_n.$$

The goal of the exercise if to show that $\mathscr{F} = \mathscr{F}(\mathscr{V})$ as subsets of $\mathscr{V}^*$.

a. Prove that $\mathscr{F} \subset \mathscr{F}(\mathscr{V})$. **Hint:** Show that it suffices to prove that, for all $n \in \mathbb{N}$, the following property $\mathscr{P}(n)$ holds

$$\mathscr{P}(n) := \left(\forall \mathscr{W} \in \mathscr{E}, \mathscr{F}_n \subset \mathscr{W}\right)$$

then prove this by induction on n.

b. Prove that $\mathscr{F}(\mathscr{V}) \subset \mathscr{F}$. **Hint:** Show that it suffices to prove that $\mathscr{F} \in \mathscr{E}$ and then prove that property.

Exercise 3. **(Bonus)** We use the same notation as in the previous exercise.

a. Construct a bijection between the set Wff of well-formed formulas and the set $\mathscr{F}$ of propositional formulas. **Hint:** Use induction on $F \in \text{Wff}$ to define a function $\varphi : \text{Wff} \to \mathscr{F}$ and induction on $n \in \mathbb{N}$ to define a function $\psi : \mathscr{F} \to \text{Wff}$.

b. Using the previous bijection between Wff and $\mathscr{F}$, show that the two definitions of the height function given in Lecture 2 indeed coincide. **Hint:** Start by figuring out what "coincide" means in the present context (for instance by using a diagram to give the expression "coincide" a precise meaning).

c. Let $w \in \mathscr{V}^*$ be an arbitrary word on $\mathscr{V}$. Show that if $\neg w \in \mathscr{F}$ then $w \in \mathscr{F}$.