Half-planes, planes, subspaces

Luisa Furthmann

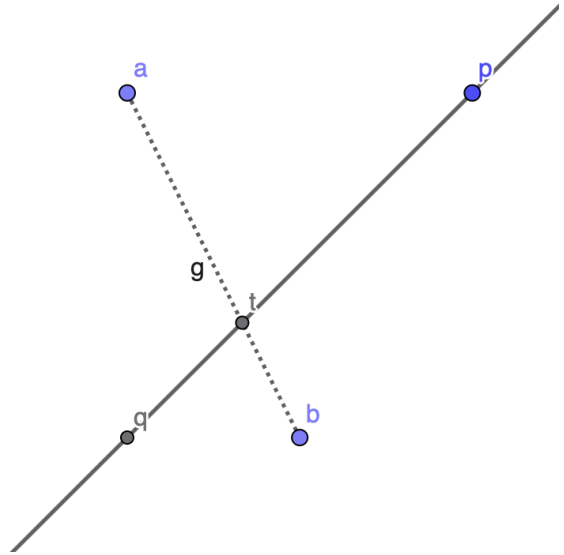
Overview

- Opposite points
- Equivalent points
- Half- planes
- Planes
- Dimensions
- supspaces

Half-planes

9.1. „a lies *opposite* to b regarding A (Ln(pq))“:

$$a|_A b: \leftrightarrow \mathbf{LnA} \wedge \boldsymbol{a \notin A} \wedge \boldsymbol{b \notin A} \wedge \exists \boldsymbol{t}\,[\boldsymbol{t \in A} \wedge \boldsymbol{Batb}]$$



9.2.

9. 5. theorem

$$\boldsymbol{a|_A c \wedge r \in A \rightarrow \forall b[Babr \rightarrow b|_A c]}$$

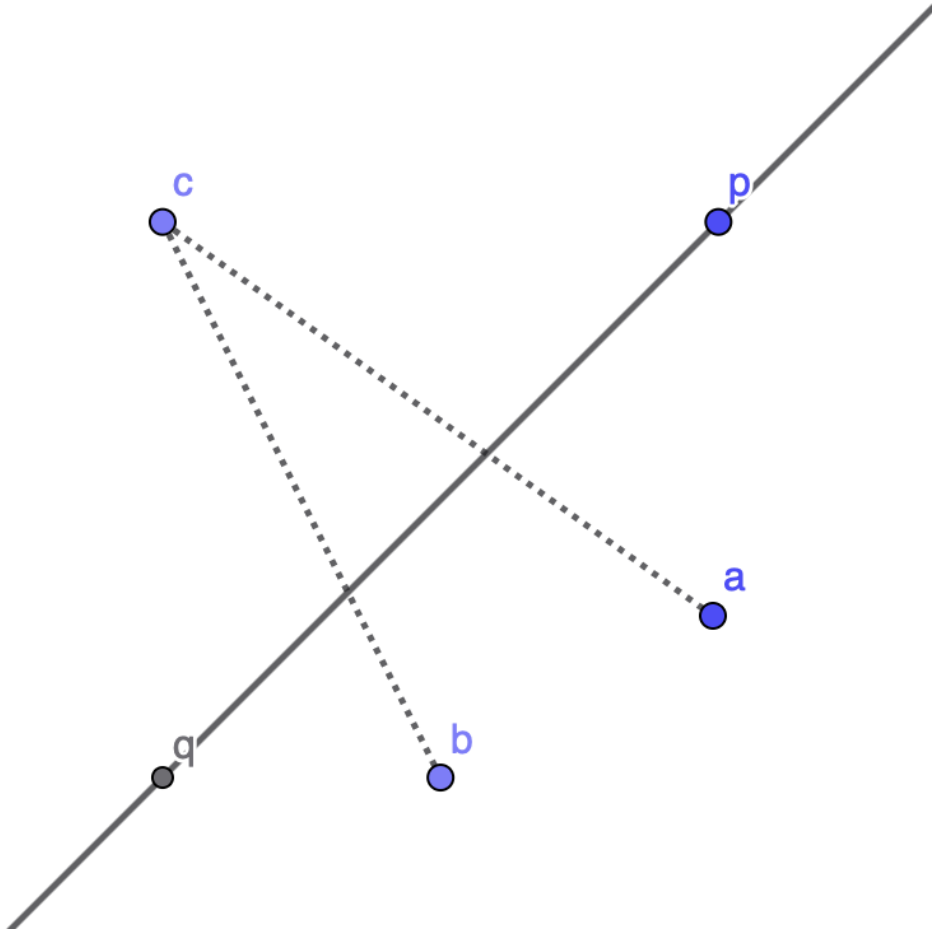
(GeoGebra animation)

9. 7. definition

- „a, b are equivalent to each other regarding to A= L(pq)“

$$a \simeq_A b: \leftrightarrow \exists c\ [a|_A c\ \wedge\ b|_A c]$$

- „a,b define the same half-plane with A as it`s limitation“



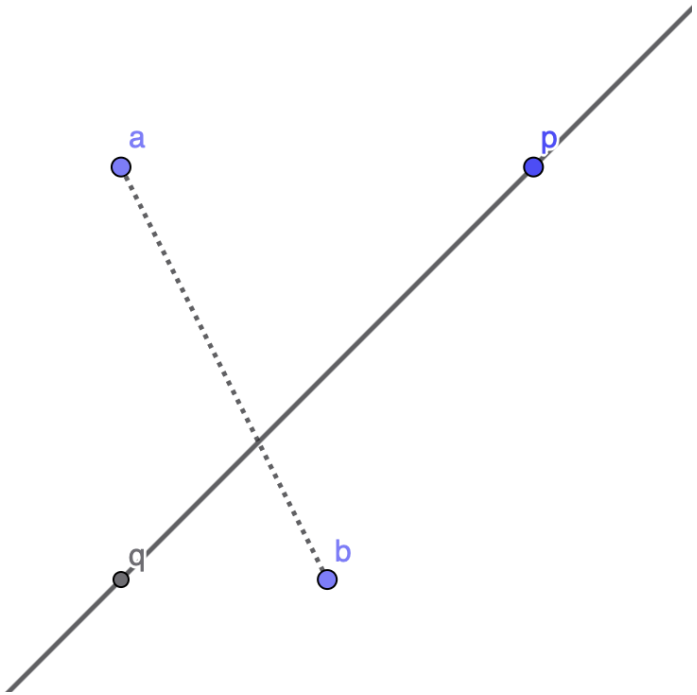
9. 8. theorem

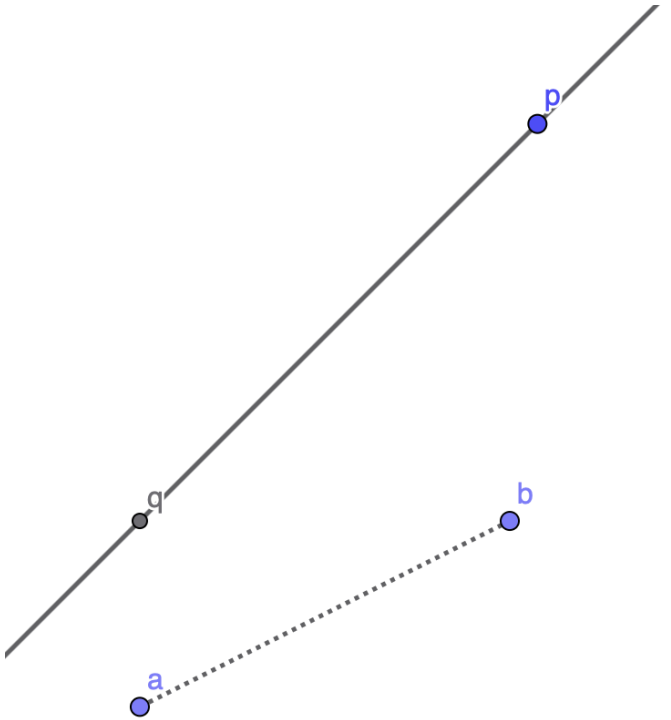
$$a|_A c \rightarrow (b|_A c \leftrightarrow a \simeq_A b)$$

(GeoGebra animation)

9. 9. theorem

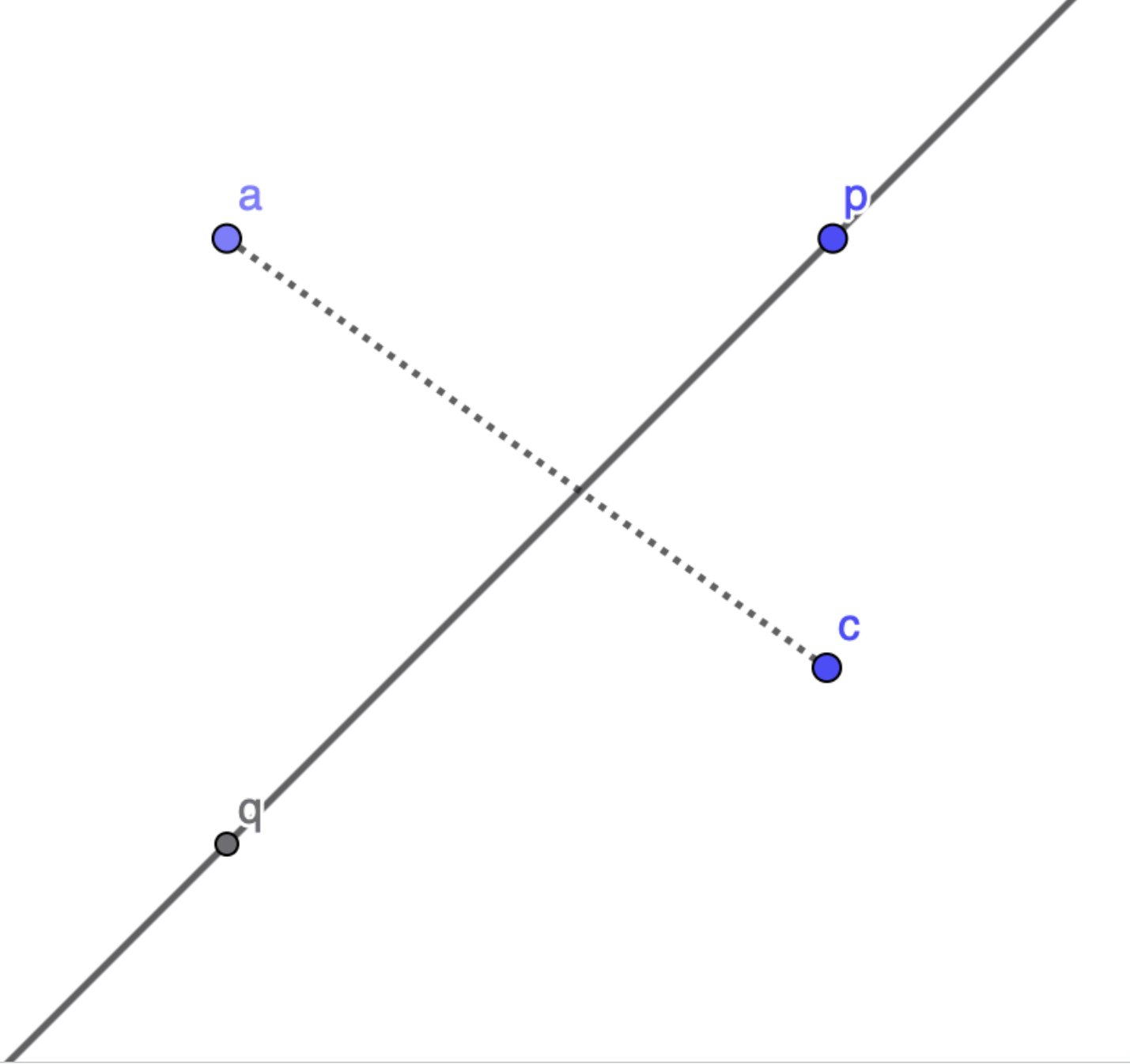
$$a|_A b \rightarrow \neg\, a \simeq_A b$$





9. 10. theorem

$$Ln\ A \wedge a \notin A \rightarrow \exists c : a|_A c$$



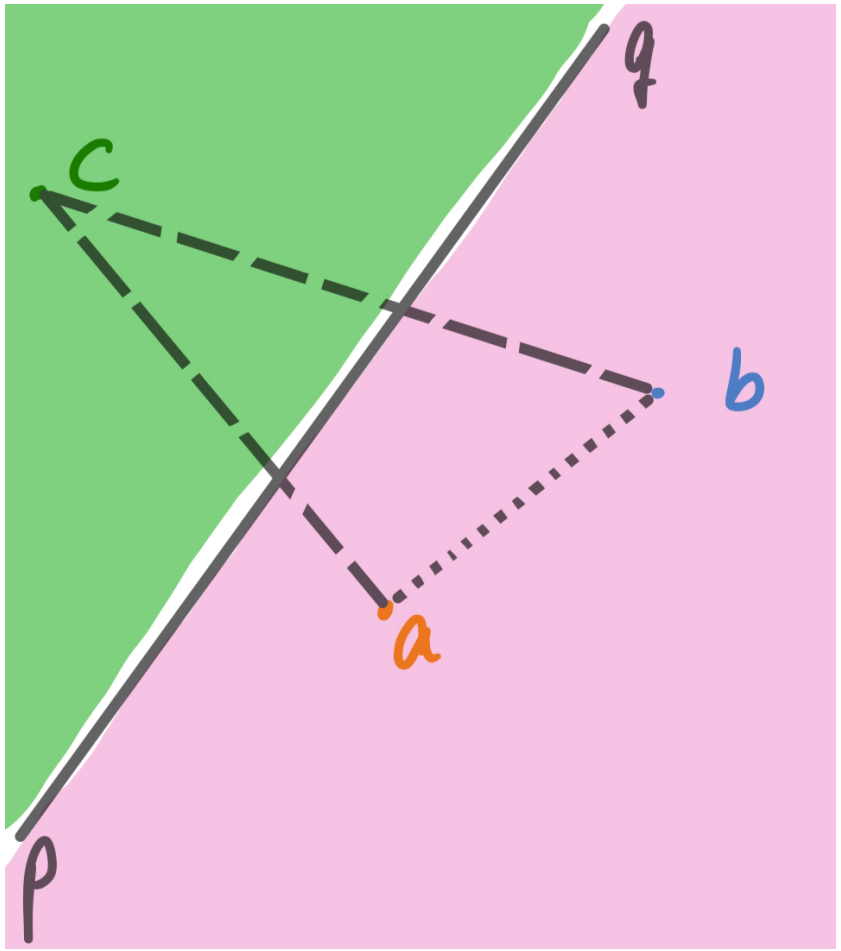
9.11., 9.12., 9.13. theorems

9.07., 9.10., 9.08. → ∀ line A: $\simeq_A$ is an equivalence relation on the set of points excluded from A, wich means:

Reflectivity: $Ln\ A\ \wedge a \notin A \rightarrow a \simeq_A a$

Symmetry: $a \simeq_A b \rightarrow b \simeq_A a$

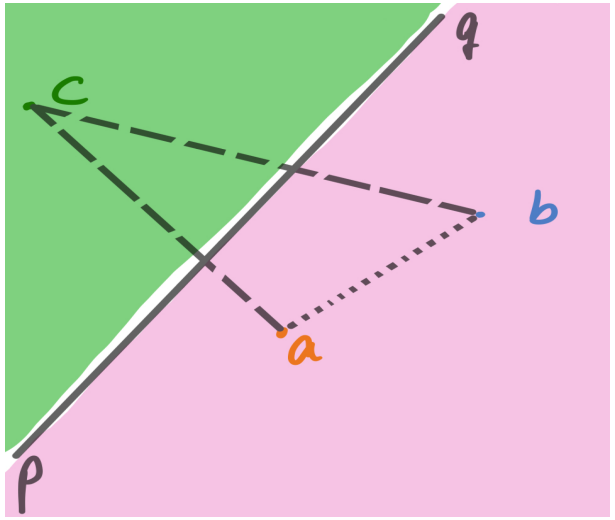
Transitivity: $a \simeq_A b\ \wedge\ b \simeq_A c \rightarrow a \simeq_A c$



9.14. definition: half-planes

- the two equivalence classes regarding to this relation are called *the **half planes** starting from A*
- The equivalence class of one ponit a is called the half-plane originating at A through a, defined by:

$$\forall Ln\ A\ \wedge a \notin A: Hp(A,a) \coloneqq \{x \mid x \simeq_A a\}$$



9.15. definition

Half-planes
$\boldsymbol{Hp(A,a)\ and\ Hp(A,b)}$
originating in A are called opposite, if $\boldsymbol{a|_A b}$ (independing from the choice of a, b)

9.17. theorem

- Half planes are *convex sets of points*

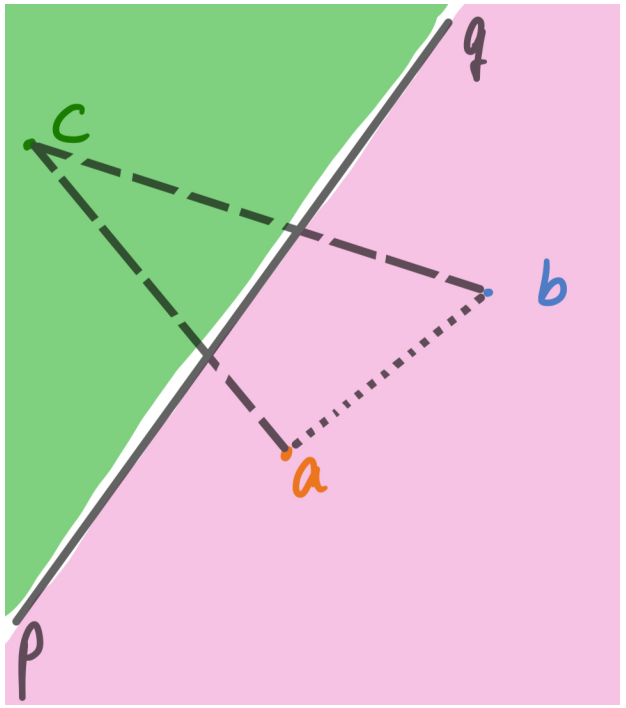
$$a \simeq_A b \ \wedge Bacb \rightarrow c \simeq_A a$$

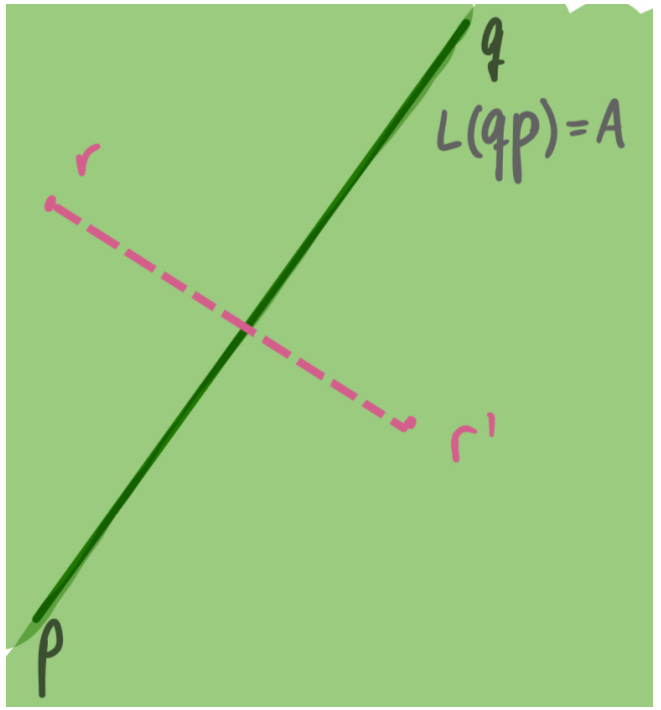
9.20. definition planes

- The plane through A and r, is defined by:

$$\forall LnA \ \wedge r \notin A \ \wedge \ r|_A r' : Pl(A,r) \coloneqq Hp(A,r) \cup A \ \cup Hp(A,r')$$

- The set of points *E* is a plane: $Pl\ E :\leftrightarrow \exists p \exists q \exists r:\ \neg Col\ pqr \ \wedge E = Pl(pqr)$

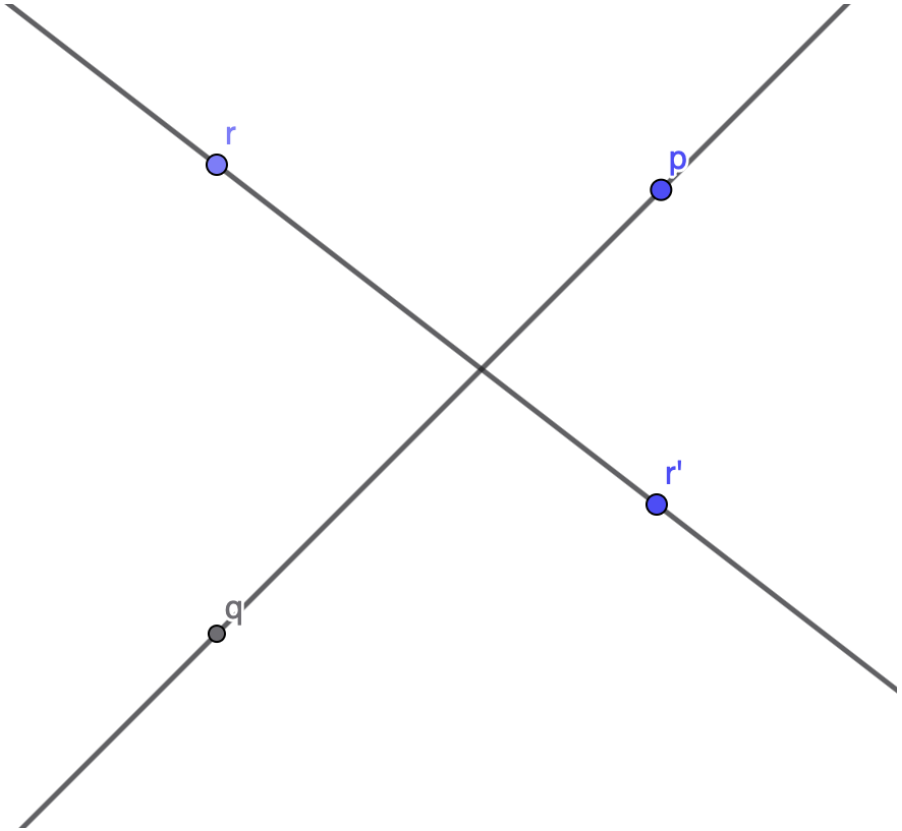




$$\boldsymbol{Pl(pqr) \coloneqq Pl\,(L(pq), r)}$$

9.23. definition

- $Pl(A, A') \coloneqq Pl(A, r)$ *defines a plane for the two intersecting lines* A, A' *with* $r \in A' \wedge r \notin A$



With 9.21 and 9.22 independent from the choice of r

9.25. theorem

- a line is totally included in a plane if it goes trought two points lying in that plane:

$$Pl\,E\ \wedge a \in E\ \wedge b \in E\ \wedge a \neq b\ \rightarrow L(ab) \subseteq E \wedge \exists c\colon E = Pl(abc)$$

(GeoGebra animation)

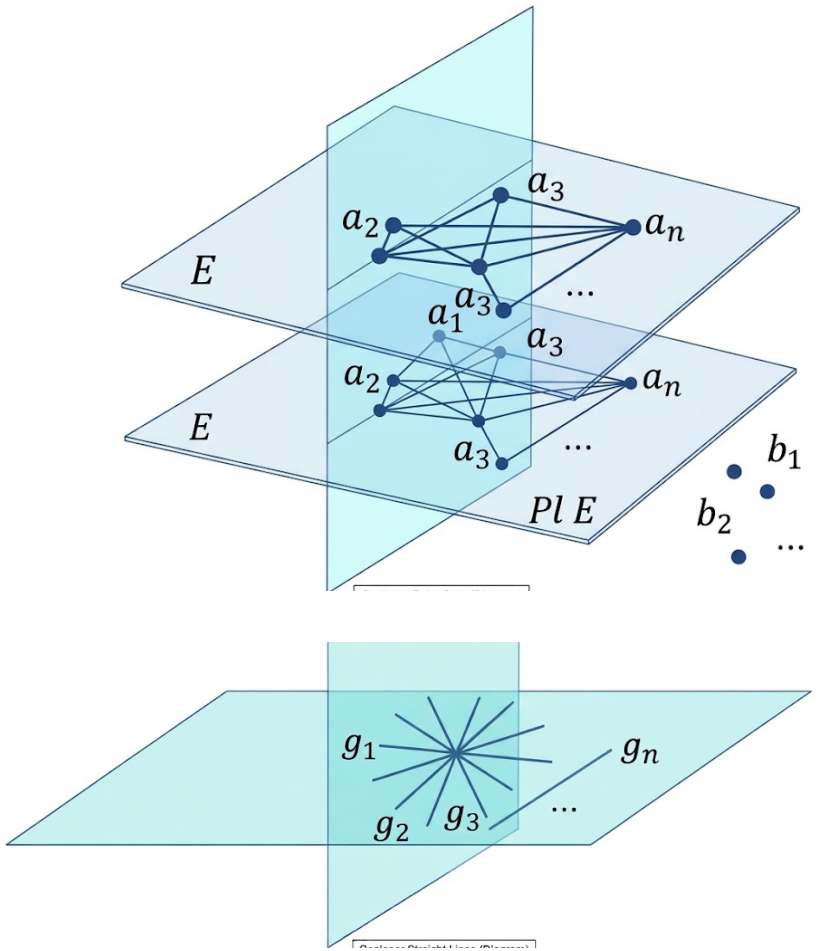
9.32. definition

- *Complanar* points are points lying in the same plane:

$$cop\ a_1 \dots a_n : \leftrightarrow\ \exists E : Pl\ E\ \wedge \bigwedge_{v=1}^{n} a_v \in E$$

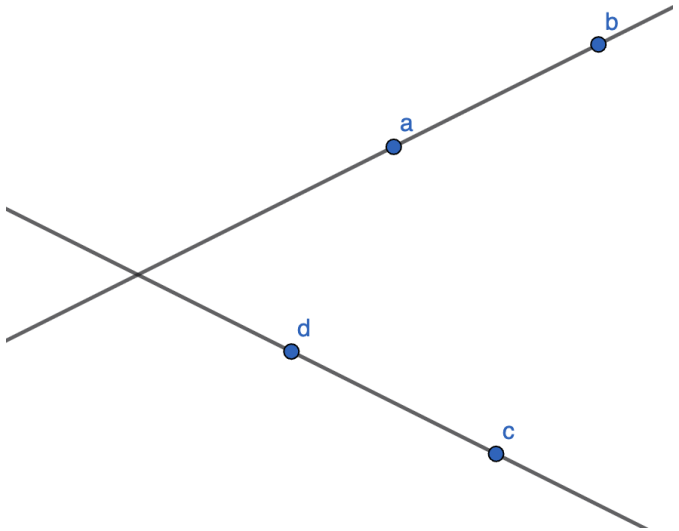
- *Complanar* lines are lines lying in the same plane:

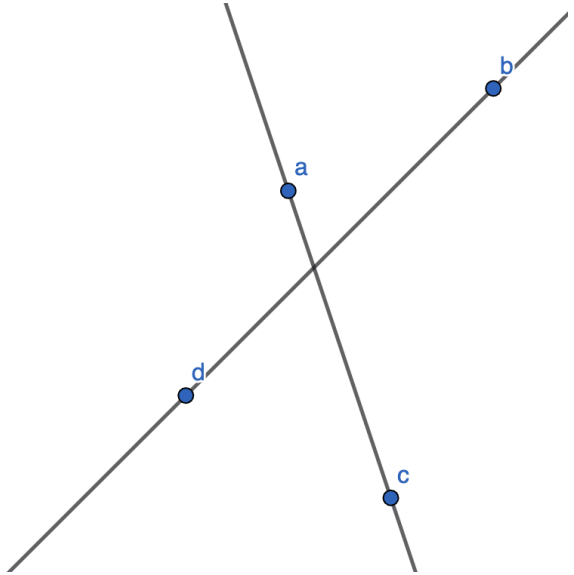
$$cop\ A_1, \dots, A_n : \leftrightarrow\ \exists E : Pl\ E \wedge \bigwedge_{v=1}^{n} (LnA_v \wedge A_v \subseteq E)$$

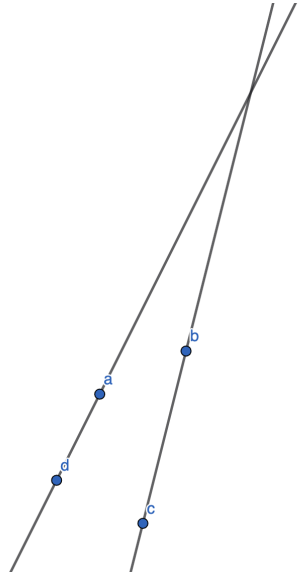


9.33. theorem

- $cop\ abcd \Leftrightarrow \exists x\{(col\ abx \wedge col\ cdx) \vee (col\ acx \wedge col\ bdx) \vee (col\ adx \wedge col\ bcx)\}$





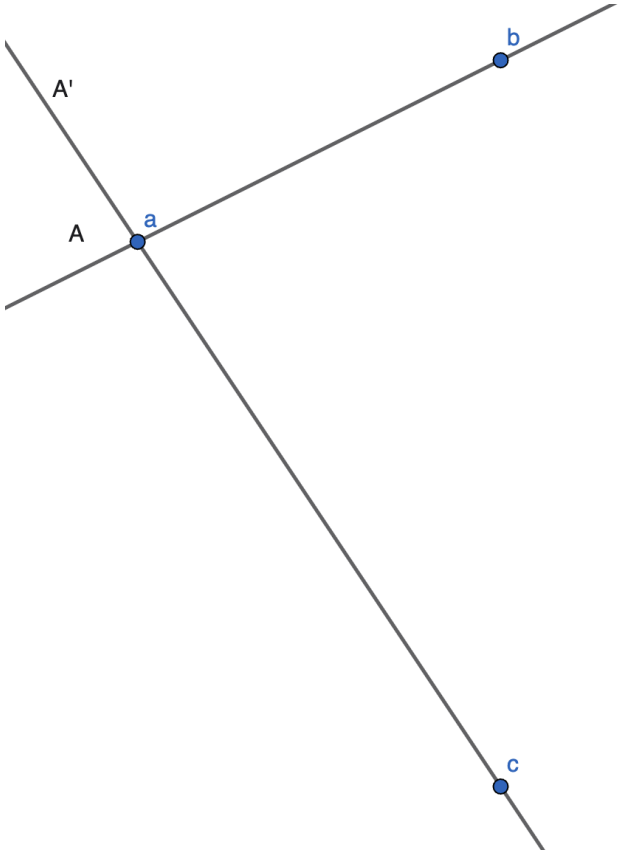


9.34. theorem

- $\neg Col\ abc \rightarrow$
  $Pl\ (abc)$ *is the smalest set of points with the following characteristics*:

1. $a, b, c \in E$ *and*
2. E *is closed with respect to the connecting lines*

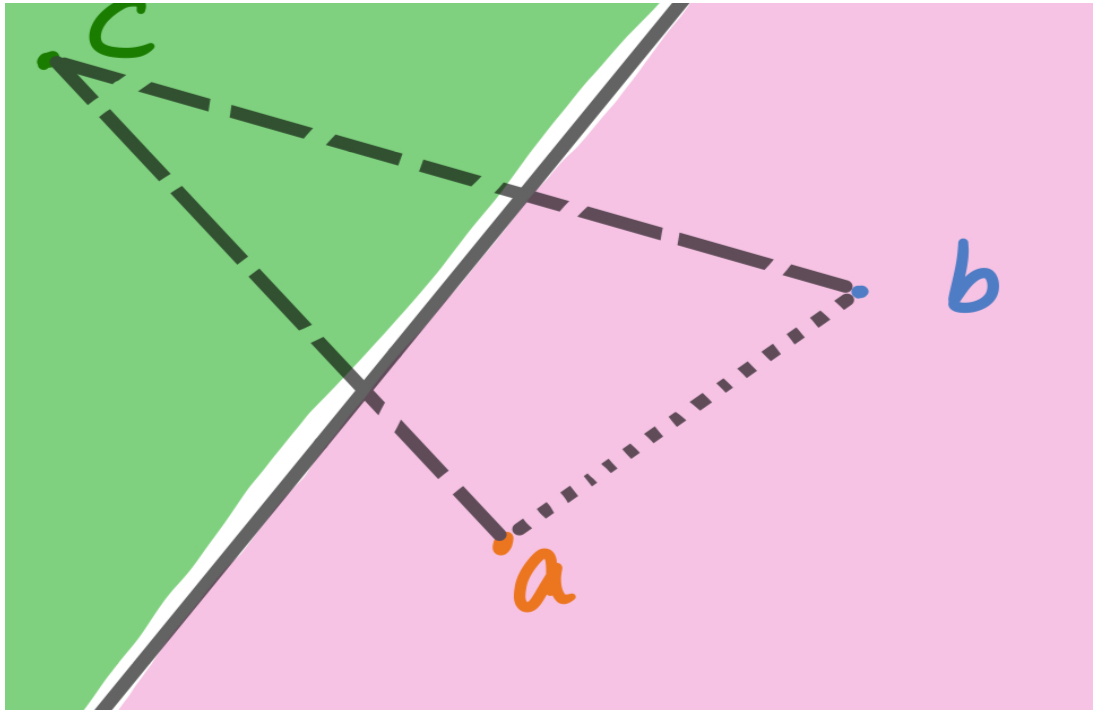
- $\Rightarrow$
  every point d *in* $Pl(abc)$ *can be constructed by two connecting lines*



Recap half planes and planes

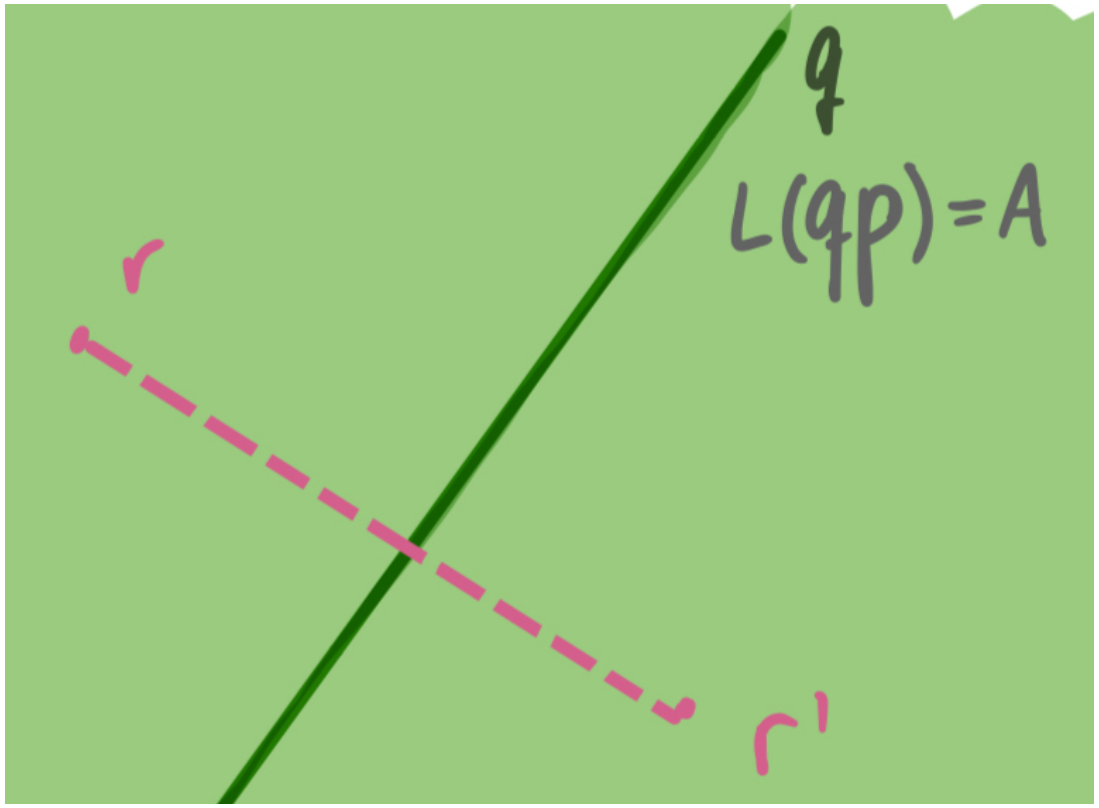
HALF-PLANE

- **OPPOSITE** POINTS REGARDING A LINE
- **EQUIVALENCE** IN TERMS OF THEIR POSITION REGARDING TO THE LINE



PLANE

- UNION OF TWO DISJOINT HALF-PLANES AND THE DIVIDING LINE



subspaces

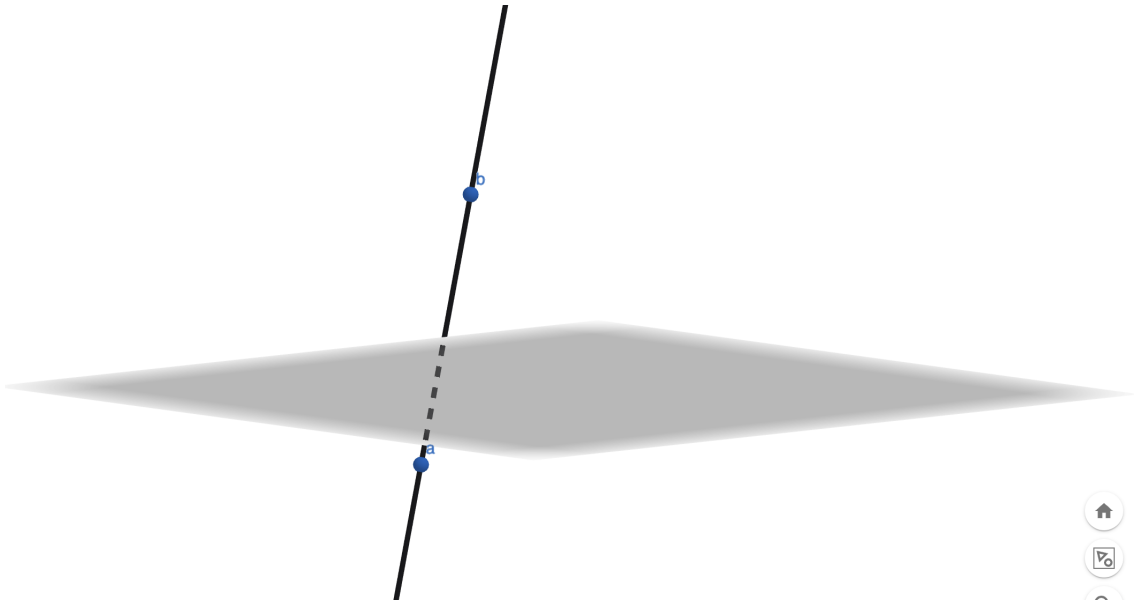
- Transfer to higher dimensions
- Dim(points)= 0
- Dim(lines)= 1
- Dim(planes)= 2

⋮

9.37. definitions

- Opposite points regarding a subspace A:

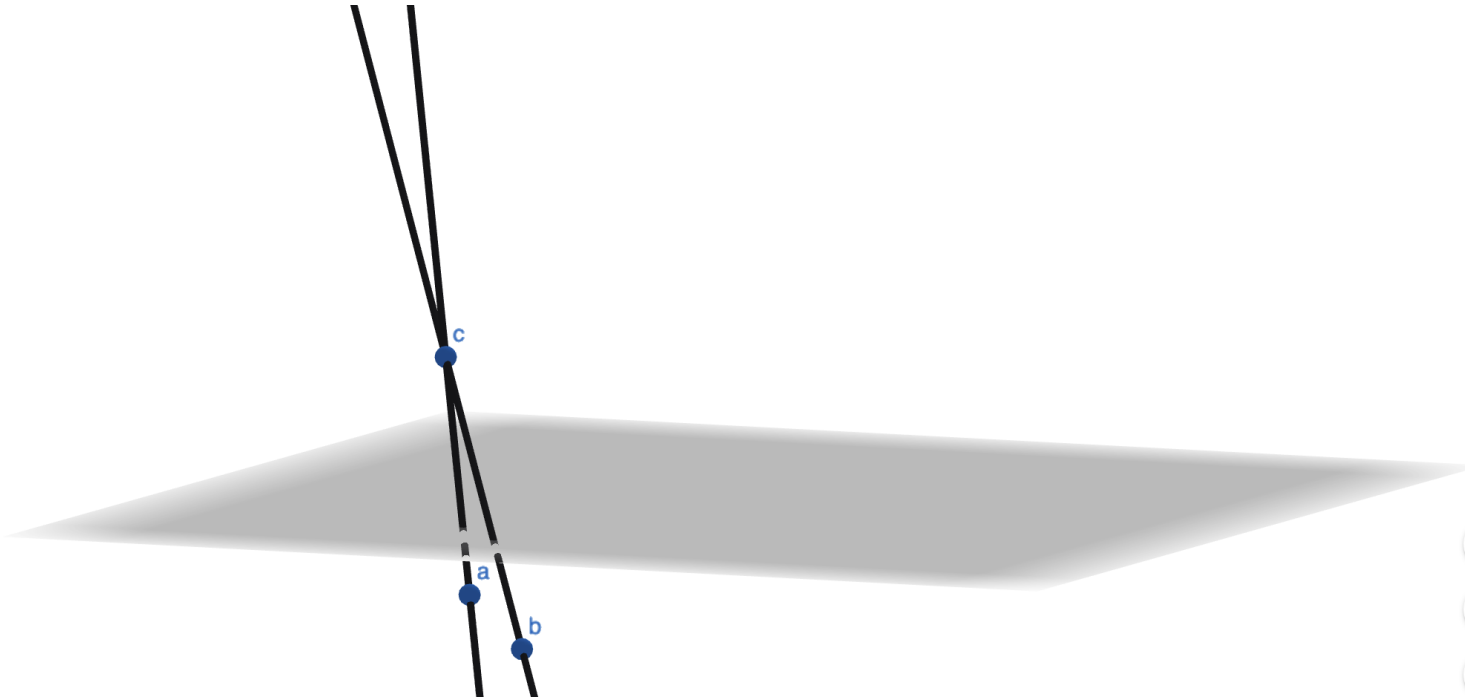
$$a|_A^n b :\leftrightarrow \mathrm{Sub}_n A \wedge a \notin A \wedge b \notin A \wedge \exists t\,[t \in A \wedge B\,a\,t\,b]$$



9.40. definition

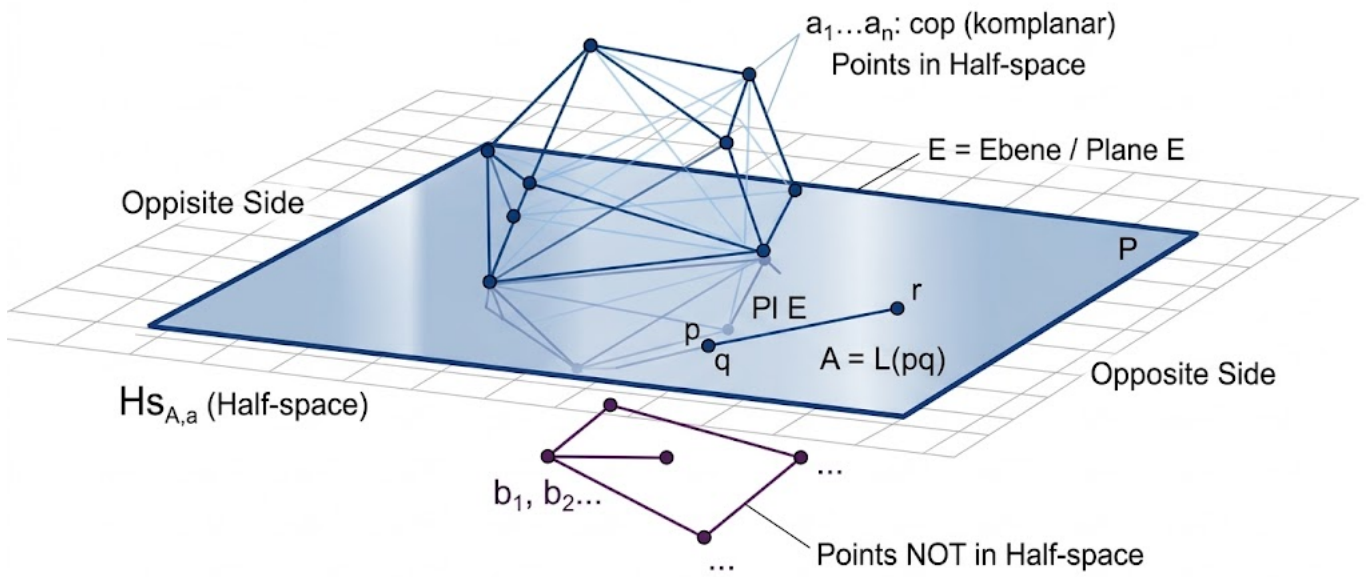
- Equvalent points regarding a subspace A

$$a \approx_A^n b :\leftrightarrow \exists c\, [a|n_A c \wedge b|n_A c]$$



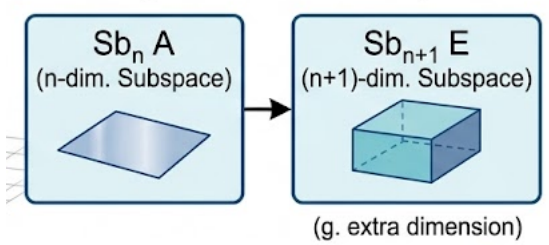
9.43. definition half-spaces

- Sub$_n A$ and $a \notin A$: $\mathrm{Hs}_{A,a} := \{x \mid x \approx_A^n a\}$
- The relation $\approx_A^n$ divides the space outside of A into exactly two disjoint half-spaces



9.44. definition subspace

- The supspace with dimension (n+1) defined by A and r:



$$\mathrm{Sb}_{n+1}(\boldsymbol{A},\boldsymbol{r}) := \mathrm{Hs}_{n+1}(\boldsymbol{A},\boldsymbol{r}) \cup \boldsymbol{A} \cup \mathrm{Hs}_{n+1}(\boldsymbol{A},\boldsymbol{r'})$$

- For n=1 this is exactly def. 9.20: half-spaces become half-planes