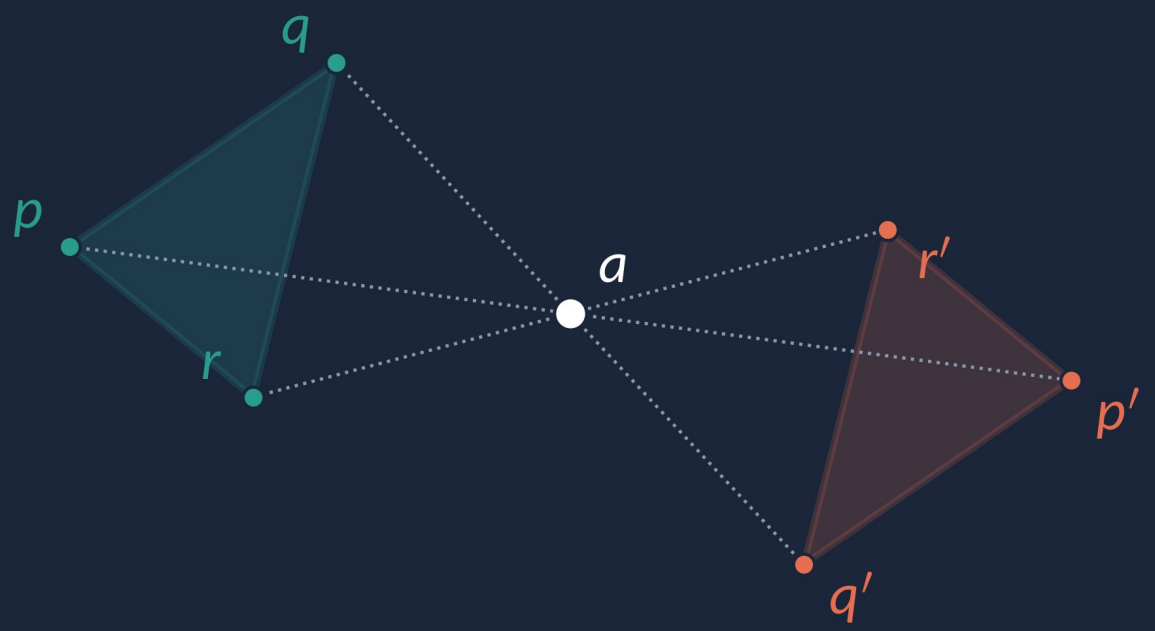
Chapters 6 & 7

Half-lines, Lines and Point Reflections

Presented by Sophia Scherer



OUTLINE

Structure of the talk

1. **Basics**

   Basic notions and axioms

2. **Chapter 6**

   Half-lines and lines

3. **Chapter 7**

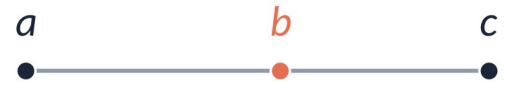
   Point reflections

4. **Applications**

   Consequences of Theorem 7.13

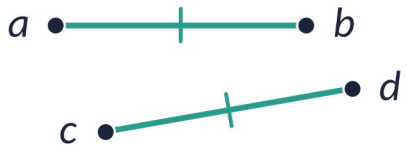
Betweenness and collinearity

Caution: B is not strict



B *abc*: *b* lies between *a* and *c*

Fig. 1



ab ≡ *cd*: segments of equal length

Fig. 2



B *pqx* **B** *pxq* **B** *xpq*

Col pqx: one of the three lies between the other two (4.10)

Fig. 3

What does not exist

- no coordinates, no numbers
- no angles, no rotations
- no lines as basic objects

Review: A4, A7

A4 Segment construction

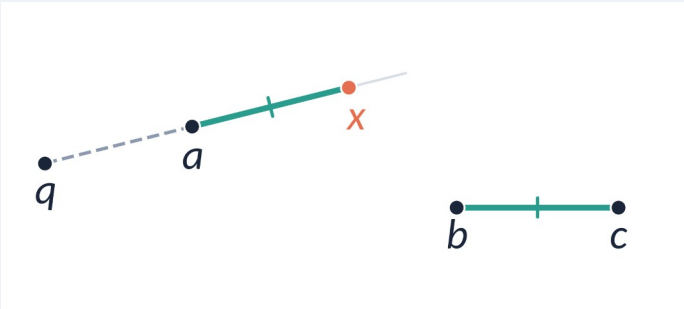


Fig. 4

for q, a, bc there is x with B qax, ax ≡ bc

A7 Pasch (inner form)

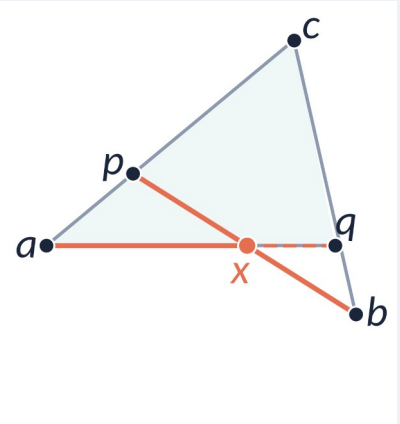


Fig. 5

intersection point – but only between existing points

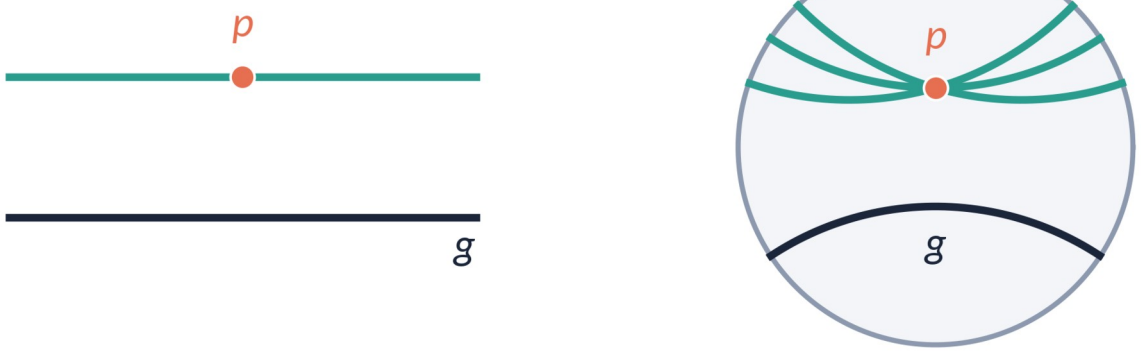
Review

A1–A3 (≡ behaves like equality)

A6 (B aba → a = b)

A8 (there are three non-collinear points)

Without the parallel axiom: **absolute geometry**



Euclidean: one parallel

hyperbolic: many parallels

Fig. 6

What this means

- Ch. 6 and 7 use only A1–A8.
- The parallel axiom A10 is not needed.
- All theorems hold in Euclidean and hyperbolic geometry.

MODEL The disk is a model of A1–A8 in which A10 is false ⇒ A10 cannot be derived from A1–A8.

GUIDING QUESTION CHAPTER 6

How do we define a ray?

Idea: ray = all points that lie "on the same side" of p.



p itself belongs to no class (relation only for points $\neq p$)

Fig. 7

Definition 6.1: Same side and opposite sides



B *apb*: opposite sides

Fig. 8



$a \underset{p}{\equiv} b$: same side

Fig. 9

(i) opposite sides

p, a, b pairwise distinct and **B**apb

(ii) same side

$a \equiv_p b \;:\Leftrightarrow\; a \neq p \wedge b \neq p \wedge [\mathbf{B}pab \vee \mathbf{B}pba]$

equivalent (6.4): Col apb ∧ ¬B apb

Def. 6.1, Thm. 6.4

Theorem 6.2: The opposite-point test



If **B***apc*: *b* on the same side as *a* ⟺ **B***bpc*

Fig. 10

$$a,b,c \neq p \wedge \mathbf{B}apc \rightarrow (\ \mathbf{B}bpc \iff a \equiv_p b\ )$$

Whatever lies on the same side as a is separated from c by p – and vice versa.

Half-lines as equivalence classes

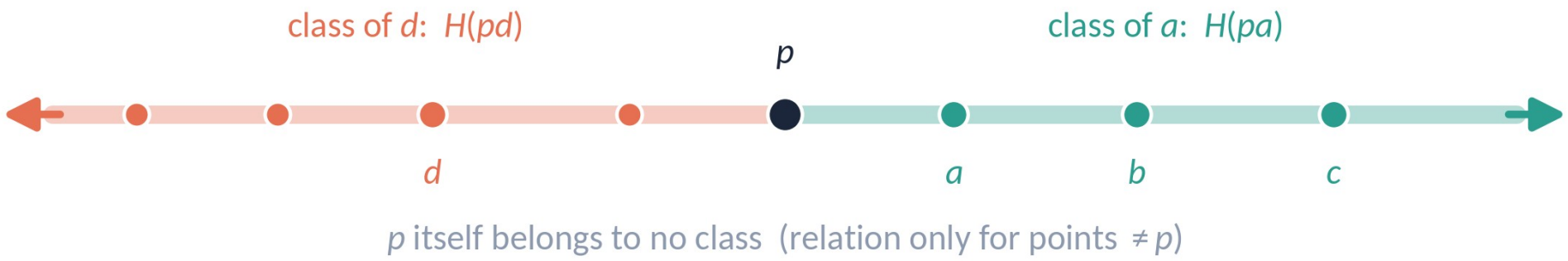


Fig. 11

$\equiv_p$ is an equivalence relation (6.5–6.7)

for a, b, c ≠ p, collinear with p:

- reflexive: $a \equiv_p a$
- symmetric: $a \equiv_p b \rightarrow b \equiv_p a$
- transitive: $a \equiv_p b \wedge b \equiv_p c \rightarrow a \equiv_p c$

Definition 6.8

$H(pa) := \{ x \mid x \equiv_p a \}$

- the initial point p does not belong to it
- every point of H(pa) is a representative
- Remark 6.10: exactly one opposite half-line

Thms. 6.5–6.7, Def. 6.8, Rem. 6.10

Definition 6.14: Lines

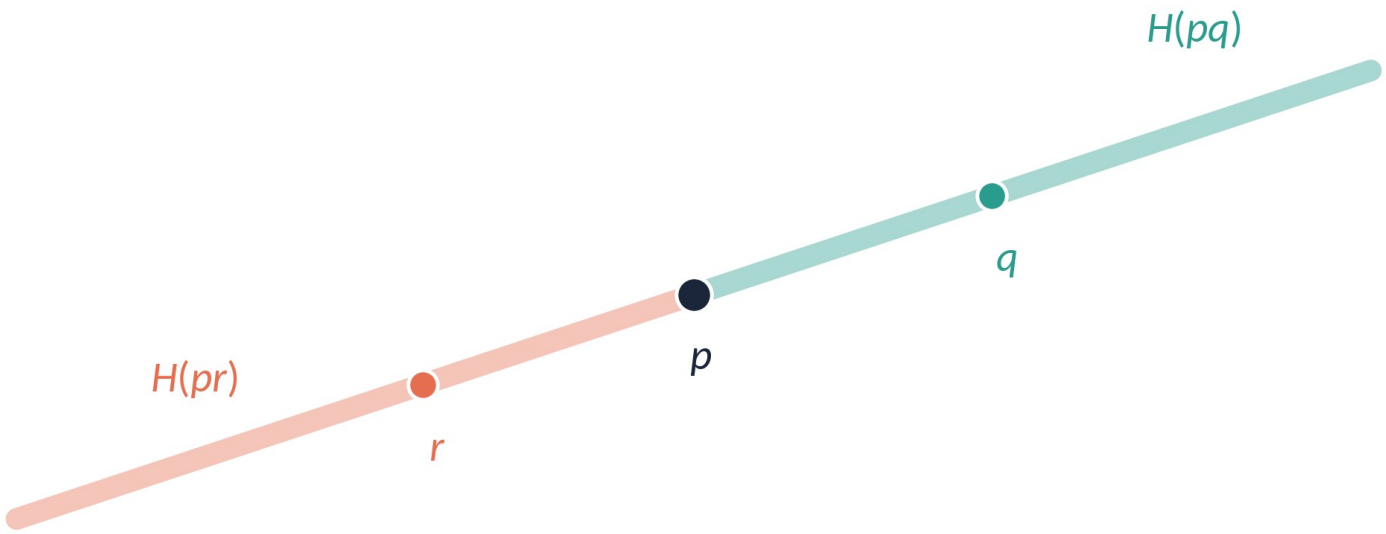


Fig. 12

The line consists of two opposite half-lines and their initial point.

Definition 6.14

L(pq) := { x | Col pqx }, p ≠ q

Theorem 6.15

Bqpr → L(pq) = H(pq) ∪ {p} ∪ H(pr)

Theorem 6.16

Every other point s ≠ p of the line determines the same line: L(pq) = L(ps).

Incidence theorems

Basis: Theorem 6.18

$a \neq b \wedge a, b \in A \rightarrow A = L(ab)$

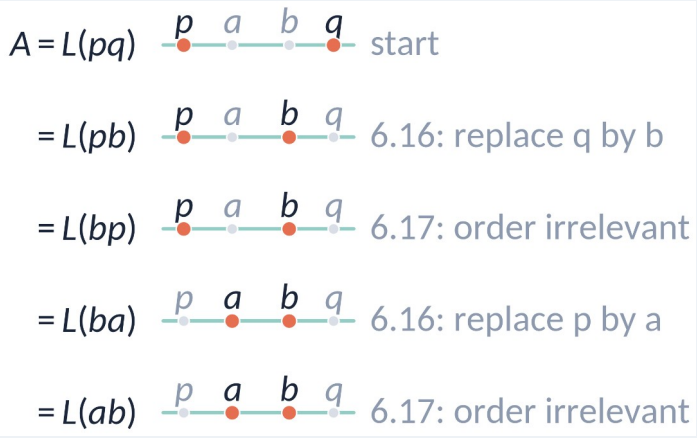


Fig. 13

Idea: replace the defining points step by step by a and b using 6.16.

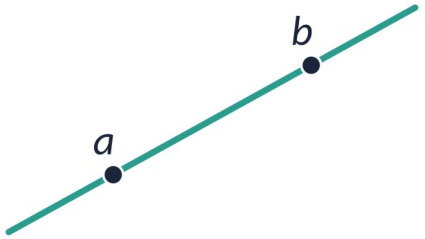


Fig. 14

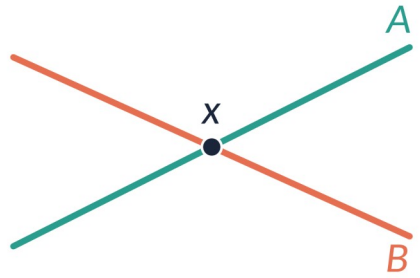


Fig. 15

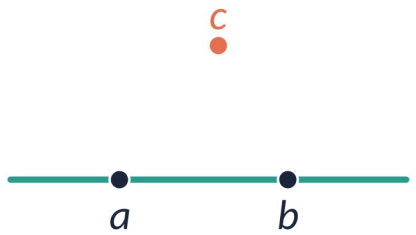


Fig. 16

Thm. 6.19 Through two points there is exactly one line.

Thm. 6.21 Two lines intersect at most once.

Thm. 6.25 For every line there is a point outside it.

REMARK 6.26

Remark 6.26: Lines are not needed

Theorem 6.18 (with lines):

$$\text{Ln } A \wedge a \neq b \wedge a \in A \wedge b \in A \rightarrow A = L(ab)$$

"If two distinct points a, b lie on a line A, then A is the line through a and b."



$A = L(pq) = L(ab)$

Fig. 17

The same statement using only points and Col:

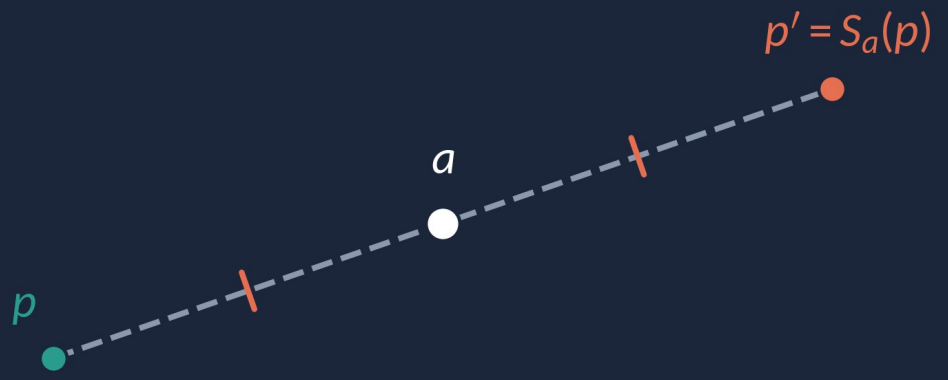
$$p \neq q \wedge a \neq b \wedge \text{Col } pqa \wedge \text{Col } pqb \rightarrow \forall x\, [\text{Col } pqx \leftrightarrow \text{Col } abx]$$

Every statement about lines can be written with points and Col alone.

SYNTAX We only talk about points, never about sets of points – Part II of the book needs this.

GUIDING QUESTION CHAPTER 7

Does a point reflection map every segment onto a congruent segment?



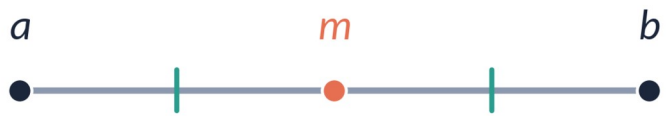
$$\mathbf{M}\,pap' :\Leftrightarrow \mathbf{B}\,pap' \wedge ap \equiv ap'$$

Fig. 18

Euclidean: "rotation by 180°". Here there are no rotations.

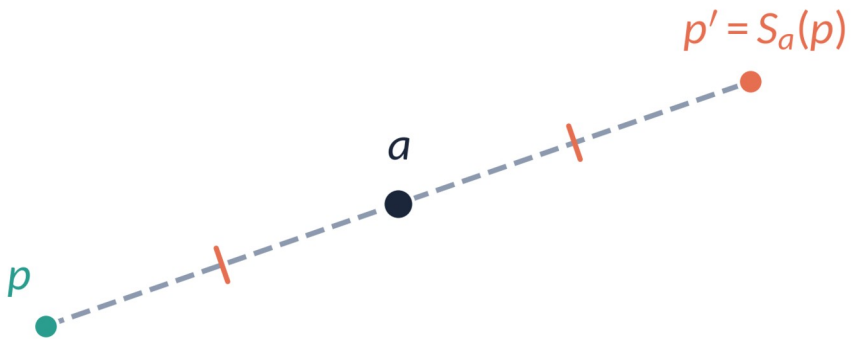
MODEL "Rotation by 180°" is an argument inside the model $\mathbb{R}^2$. Here we need a proof from the axioms.

Midpoint and mirror image



M*amb*: **B***amb* and $ma \equiv mb$

Fig. 19



$\mathbf{M}\,pap' :\Leftrightarrow \mathbf{B}\,pap' \wedge ap \equiv ap'$

Fig. 20

Definition 7.1 Midpoint

Mamb :⟺ **B**amb ∧ ma ≡ mb

also: reflection in m maps a to b

Theorem 7.4 Mirror image

For every p there is exactly one p′ with M pap′.

Existence: A4 · uniqueness: 2.12

Axiom A4 (existence)

∃x (**B**qax ∧ ax ≡ bc)

here: extend pa beyond a by ap

Definition 7.5 Point reflection

We write $p' = S_a(p)$ (mirror image of p in a). The map S_a is the point reflection in a.

Properties of point reflections

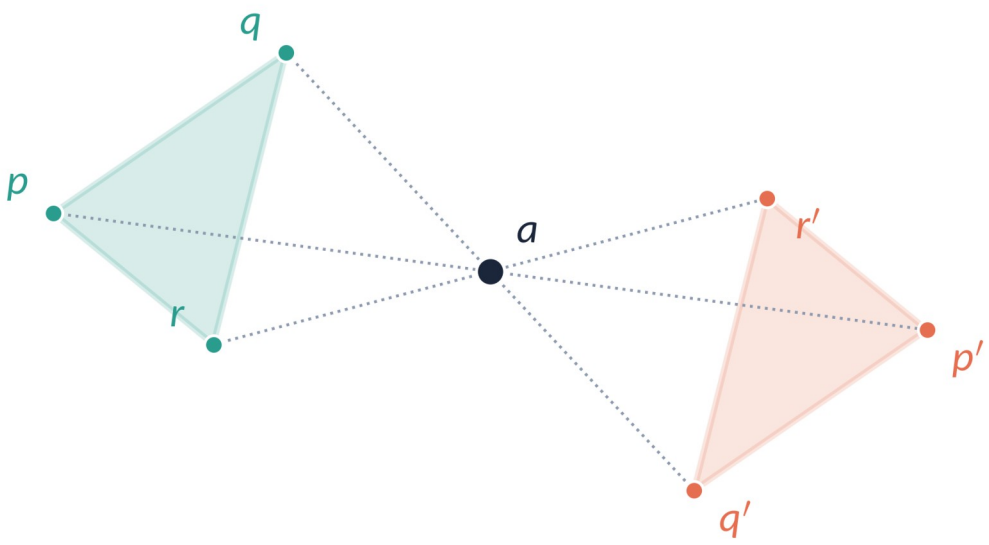


Fig. 21

Thm. 7.7

$S_a S_a p = p$

reflecting twice = identity

Thm. 7.10

$S_a p = p \leftrightarrow p = a$

only fixed point: a

Thm. 7.11

S_a bijective

one-to-one onto the space

Thm. 7.12

S_a involution

not id, but twice it is

Open question: Does S_a also map every segment onto a congruent segment? → Theorem 7.13

Theorem 7.13: Every segment is congruent to its image

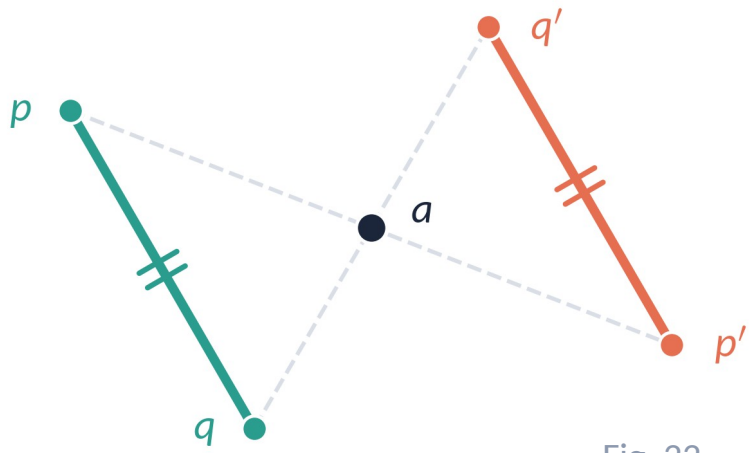


Fig. 22

Theorem 7.13

$pq \equiv S_a(p)\, S_a(q)$

Corollary 7.14

S_a is a motion, hence an automorphism: it preserves **B** and ≡ (7.15, 7.16) – and therefore also midpoints.

SYNTAX The proof uses only A1–A8 ⇒ Theorem 7.13 holds in every model, Euclidean and hyperbolic.

Five-segment statements

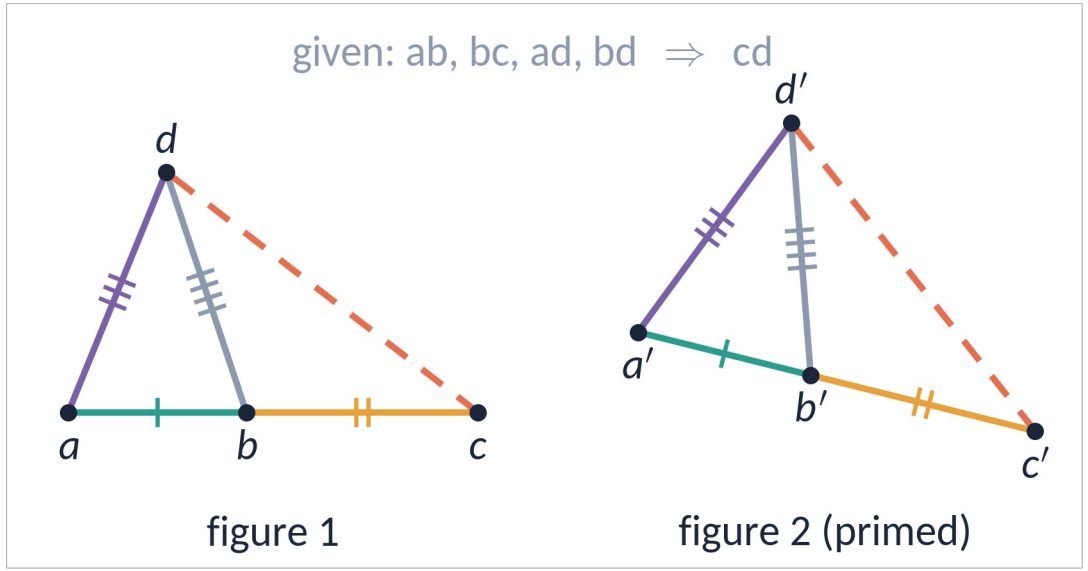


Fig. 23

OFS + A5: outer segment cd

B abc, B a′b′c′, ab ≡ a′b′, bc ≡ b′c′,
ad ≡ a′d′, bd ≡ b′d′, a ≠ b ⇒ cd ≡ c′d′

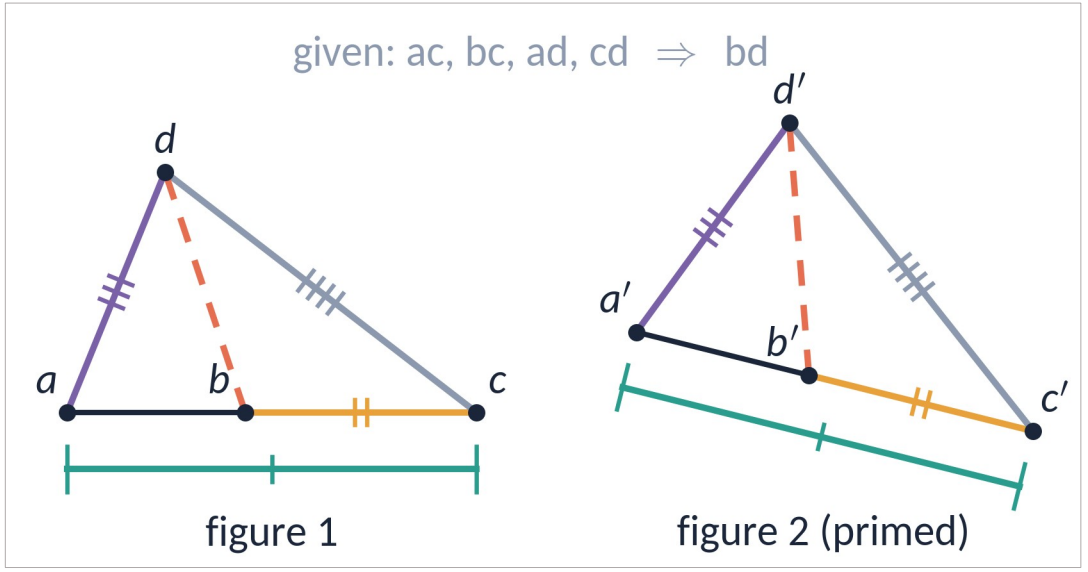


Fig. 24

IFS + Theorem 4.2: inner segment bd

B abc, B a′b′c′, ac ≡ a′c′, bc ≡ b′c′,
ad ≡ a′d′, cd ≡ c′d′ ⇒ bd ≡ b′d′

Constructing the frame (A4)

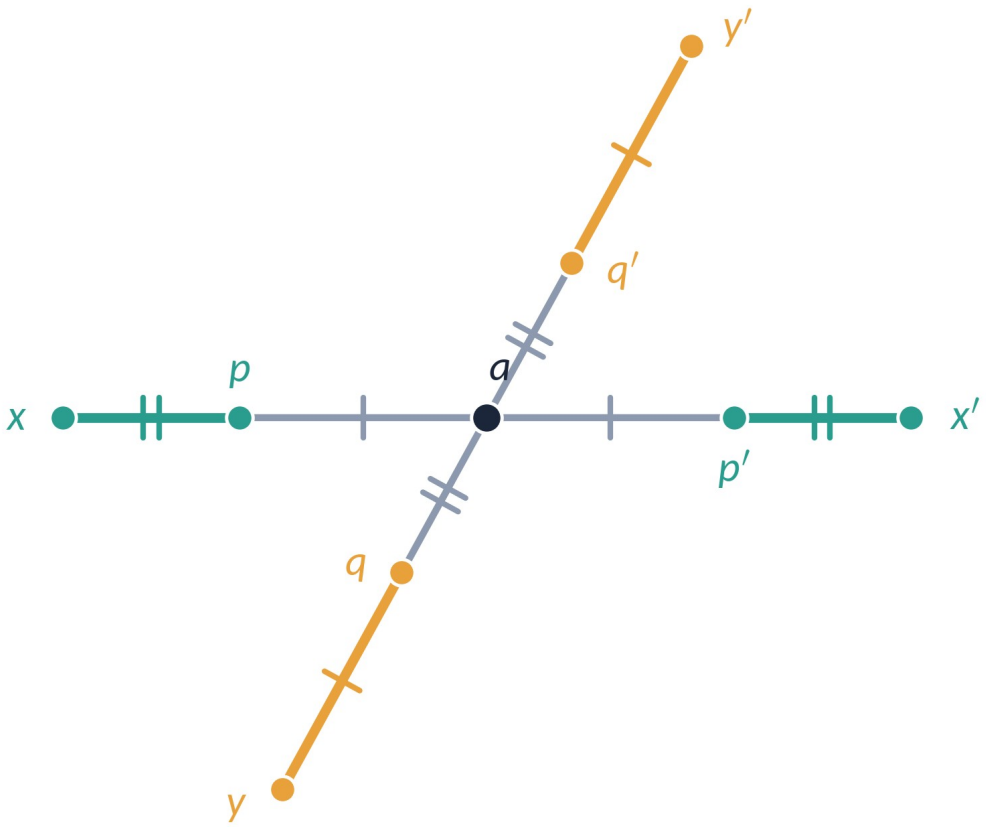


Fig. 25

Choose x, y, x′, y′ with

x: **B**p′px ∧ px ≡ qa

y: **B**q′qy ∧ qy ≡ pa

x’: **B**xp′x′ ∧ p′x′ ≡ qa

y’: **B**yq′y′ ∧ q′y′ ≡ pa

2.11 ⇒ ax ≡ ay ≡ ay′ ≡ ax′

Tick marks: | ≡ ap || ≡ aq

Proof 7.13, Fig. 17

Outer sides of the frame (A5)

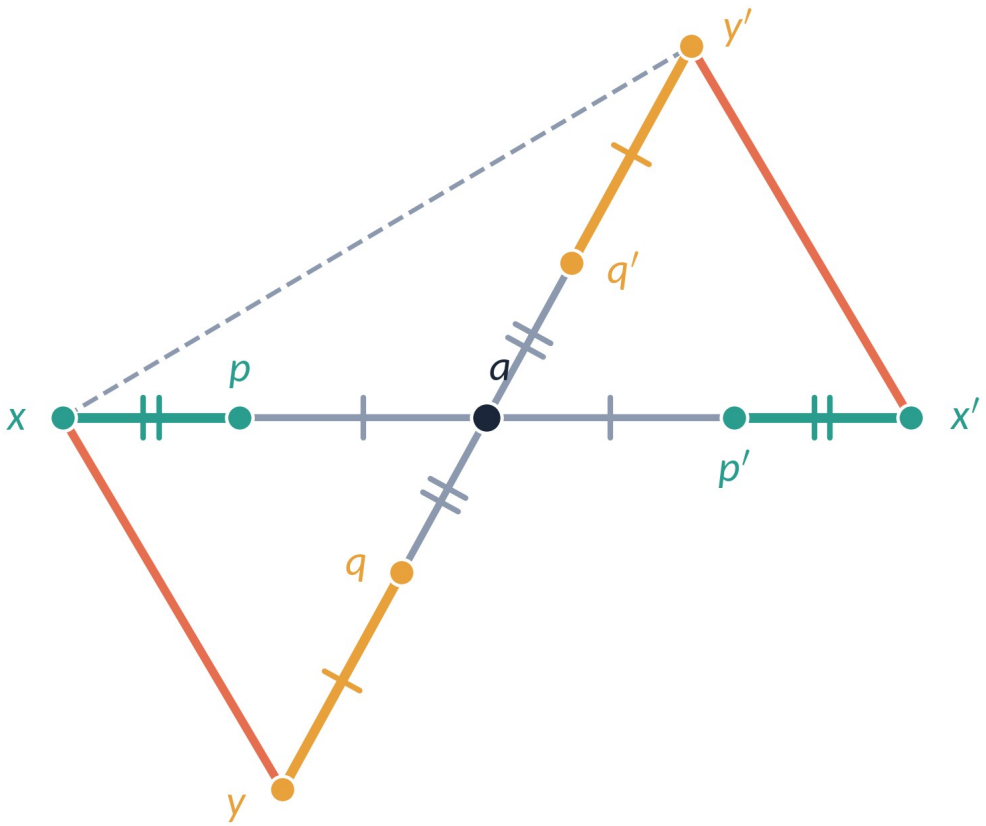


Fig. 26

OFS (x a x′ y′ / y′ a y x):

Bxax′ and **B**y′ay (step 1)

xa ≡ y′a (step 1)

ax′ ≡ ay (step 1)

xy′ ≡ y′x (same segment)

ay′ ≡ ax (step 1)

x ≠ a, since p ≠ a and **B**apx

A5 ⇒ x′y′ ≡ xy

Proof 7.13, Fig. 17

Inner segment qx (Theorem 4.2)

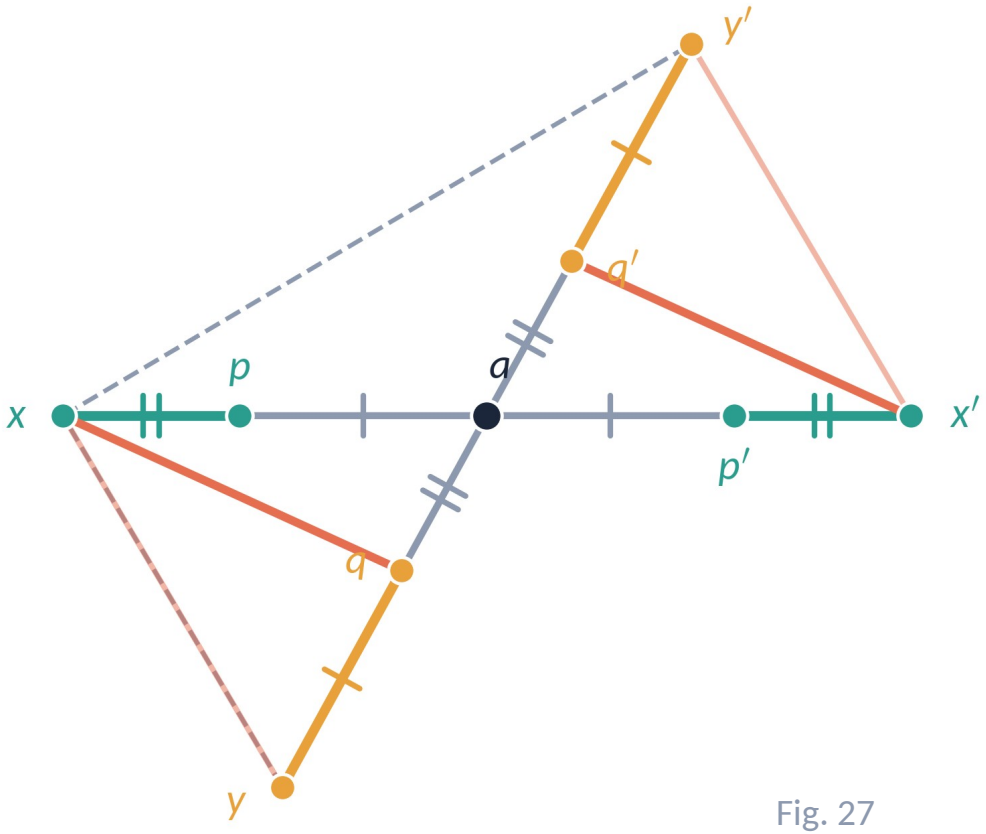


Fig. 27

IFS (y q a x / y′ q′ a x′):

Byqa (step 1)

By′q′a (step 1)

ya ≡ y′a (step 1)

qa ≡ q′a (by definition of q’)

yx ≡ y′x′ (step 2)

ax ≡ ax′ (by step 1)

4.2 ′ qx ≡ q′x□

Proof 7.13, Fig. 17

Inner segment pq (Theorem 4.2)

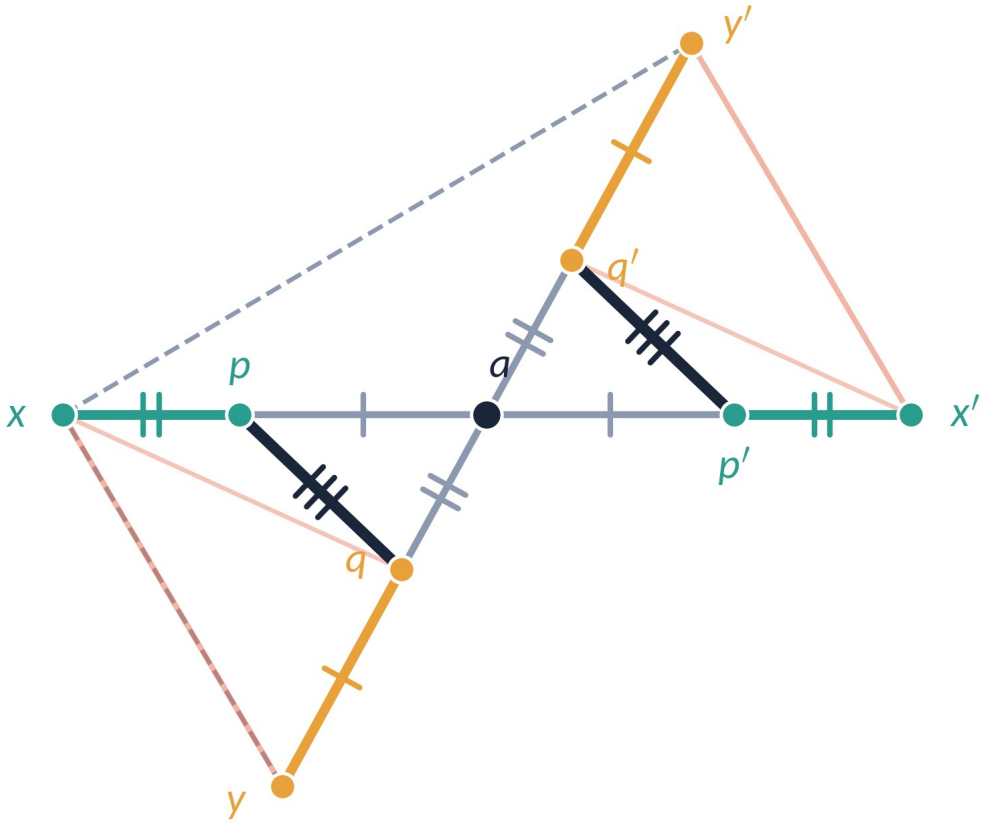


Fig. 28

IFS (x p a q / x′ p′ a q′):

Bxpa (step 1)

Bx′p′a (step 1)

xa ≡ x′a (step 1)

pa ≡ p′a (by definition of p')

xq ≡ x′q′ (step 3)

aq ≡ aq′ (by definition of p')

Thm. 4.2 ⇒ pq ≡ p′q′ □

Proof 7.13, Fig. 17

Theorem 7.17: Uniqueness of the midpoint

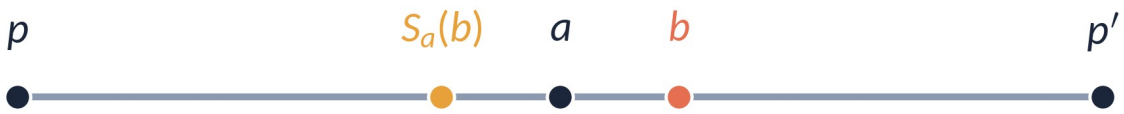


Fig. 29

$$\mathbf{M}pap' \wedge \mathbf{M}pbp' \rightarrow a = b$$

Why? (idea, using 7.13)

- Every segment is congruent to its image under S_a (7.13).
- So $pb \equiv p'S_a(b)$ and $p'b \equiv pS_a(b)$: the roles of p and p′ are swapped.
- By 4.19: $S_a(b) = b$. The only fixed point is a, so b = a.

Existence of a midpoint: only Theorem 8.22.

Theorem 7.19: Composition of reflections

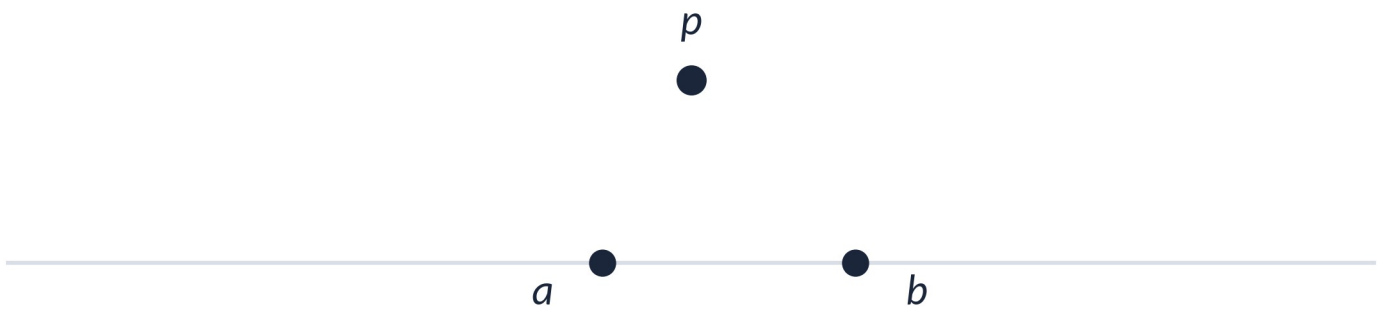


Fig. 30

Prediction

Reflect first in b, then in a –
is that the same as first in a, then in b?

Vote: yes or no?

Theorem 7.19: Composition of reflections

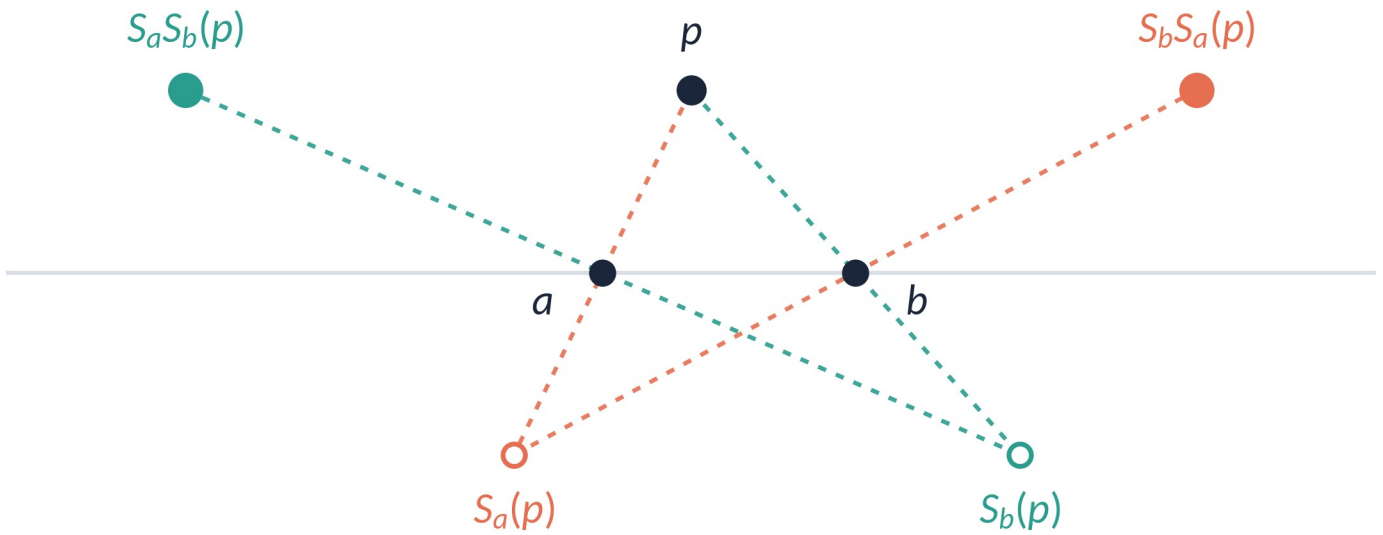


Fig. 31

$$S_aS_bp = S_bS_ap \iff a = b$$

Intuitively: two translations by twice ab, in opposite directions.

Answer

Reflect first in b, then in a –
is that the same as first in a, then in b?

No – only if a = b.

Proof with 7.14 and 7.17 (in the book).

Outlook: further theorems of Chapter 7

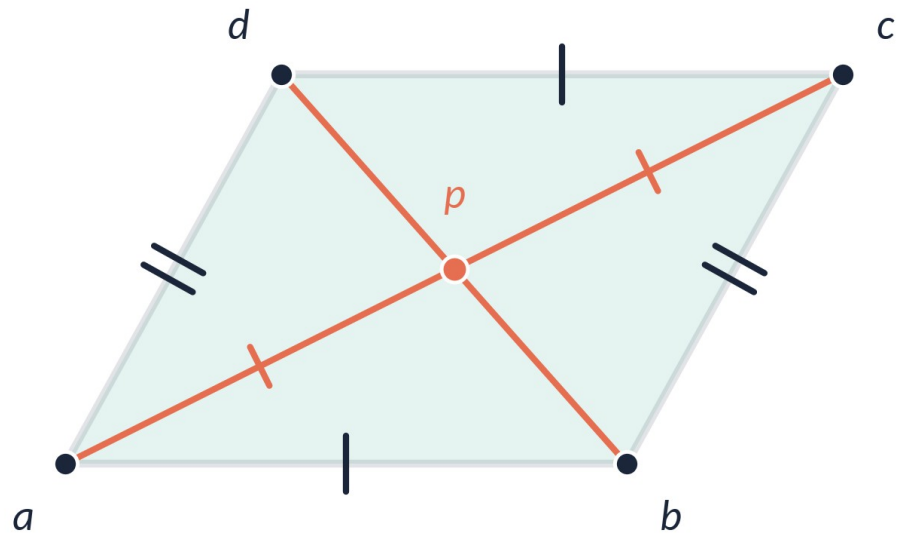


Fig. 32

7.18 $S_a p = S_b p \to a = b$

7.20 Col amb $\wedge$ ma $\equiv$ mb $\to$ a = b or M amb

7.25 ca $\equiv$ cb $\Rightarrow$ ab has a midpoint (special case of 8.22)

7.21 Symmetric quadrilateral: opposite sides congruent $\Rightarrow$ diagonals bisect each other. Without the parallel axiom, no "parallelogram".

Conclusion

- **Half-line** = equivalence class, governed by the test-point theorem 6.2
- **Line** = two half-lines + point; Hilbert's axioms become theorems
- **Point reflection** = motion; proof via frame + five-segment statements
- **Method** = deduction from the axioms (syntax); models show what cannot be deduced → Part II of the book

References

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- Kasten, H.; Vogel, D. (2018): *Grundlagen der ebenen Geometrie*. Berlin: Springer Spektrum.
- Kasten, H.; Vogel, D. (2025): *Einführung in die Geometrie*. Lecture notes (in German), summer term 2025, Heidelberg University.