First results on betweenness and congruence

Definition 2.6

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- a and b are **endpoints**
- ab is called the **connecting segment** of points a and b
- a=b the line segment is a **zero segment**
- All other segments are called **non-degenerate segments**

Definition 2.10

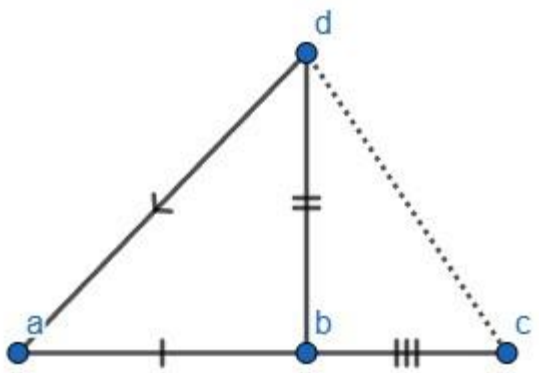
- The points a, b, c, d, a′, b′, c′, d′ form a Outer-Five-Segment-Configuration:

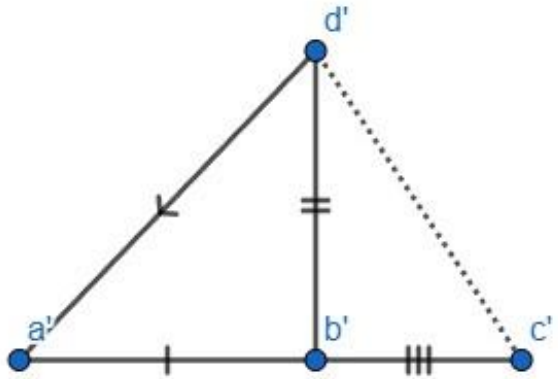
  $\mathrm{AFS}\begin{pmatrix} a & b & c & d \\ a' & b' & c' & d' \end{pmatrix} :\leftrightarrow Babc \wedge Ba'b'c' \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge$
  $ad \equiv a'd' \wedge\ bd \equiv b'd'$

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$$\mathrm{AFS}\begin{pmatrix} a & b & c & d \\ a' & b' & c' & d' \end{pmatrix} :\leftrightarrow \mathrm{Babc} \wedge \mathrm{Ba'b'c'} \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge ad \equiv a'd' \wedge bd \equiv b'd'$$





Definition 2.10

- We can now shorten A_5 to:

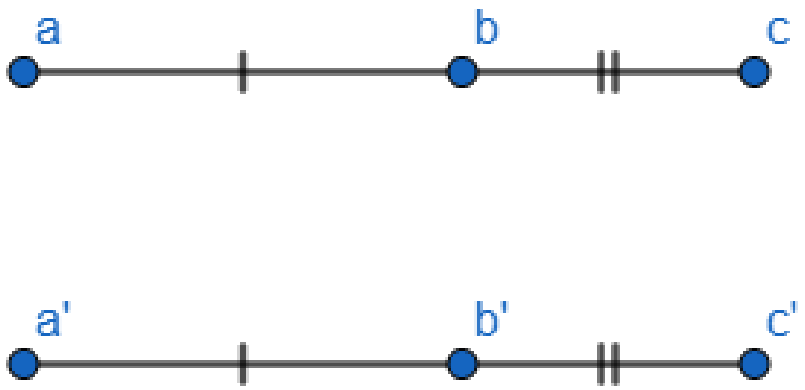
$$\mathrm{AFS}\begin{pmatrix} \mathrm{a} & b & c & d \\ \mathrm{a}' & b' & c' & d' \end{pmatrix} \wedge\ \mathrm{a} \neq \mathrm{b} \rightarrow \mathrm{cd} \equiv\ \mathrm{c}'\mathrm{d}'$$

Theorem 2.11

- $Babc \wedge Ba'b'c' \wedge ab \equiv a'b' \wedge bc \equiv b'c' \rightarrow ac \equiv a'c'$

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Case 1: a = b

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Case 1: a = b

- $\mathrm{Babc} \wedge \mathrm{Ba'b'c'} \wedge \mathrm{ab} \equiv \mathrm{a'b'} \wedge \mathrm{bc} \equiv \mathrm{b'c'}$

  $\mathrm{a} = \mathrm{b} \overset{A_3}{\longrightarrow} \mathrm{a'} = \mathrm{b'}$

Case 1: a = b

- $Babc \wedge Ba'b'c' \wedge ab \equiv a'b' \wedge bc \equiv b'c'$

  $a = b \overset{A_3}{\rightarrow} a' = b'$

  $a = b \wedge a' = b' \rightarrow ac \equiv a'c'$

Case 2: $a \neq b$

- $\mathrm{AFS}\begin{pmatrix} a & b & c & a \\ a' & b' & c' & a' \end{pmatrix} \overset{A_5}{\rightarrow} ac \equiv a'c'$

■

Definition 3.8

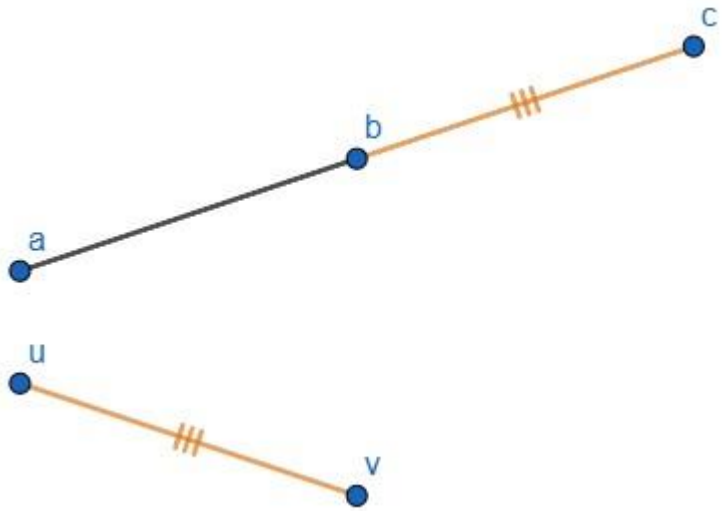
- the arrangement of n points $a_1, \dots, a_n$ on a line can be described by a **generalized intermediate** *relation* $\boldsymbol{B_n}$
- $B_n(a_1, a_2, \dots, a_n) \leftrightarrow \bigwedge_{1 \le i < j < k \le n} B\left(a_i, a_j, a_k\right);\ (n \geq 3)$

Theorem 3.14

- $\exists c[Babc \wedge b \neq c]$

Theorem 3.14

- ∃c[Babc ∧ b ≠ c]



- <u>Proof:</u>

  -Let u, v be two different points (as in 3.13)

  -Let c be a point with *Babc ∧ bc ≡ uv (A4)*

  -c satisfies the condition (by A3)

Definition 4.1

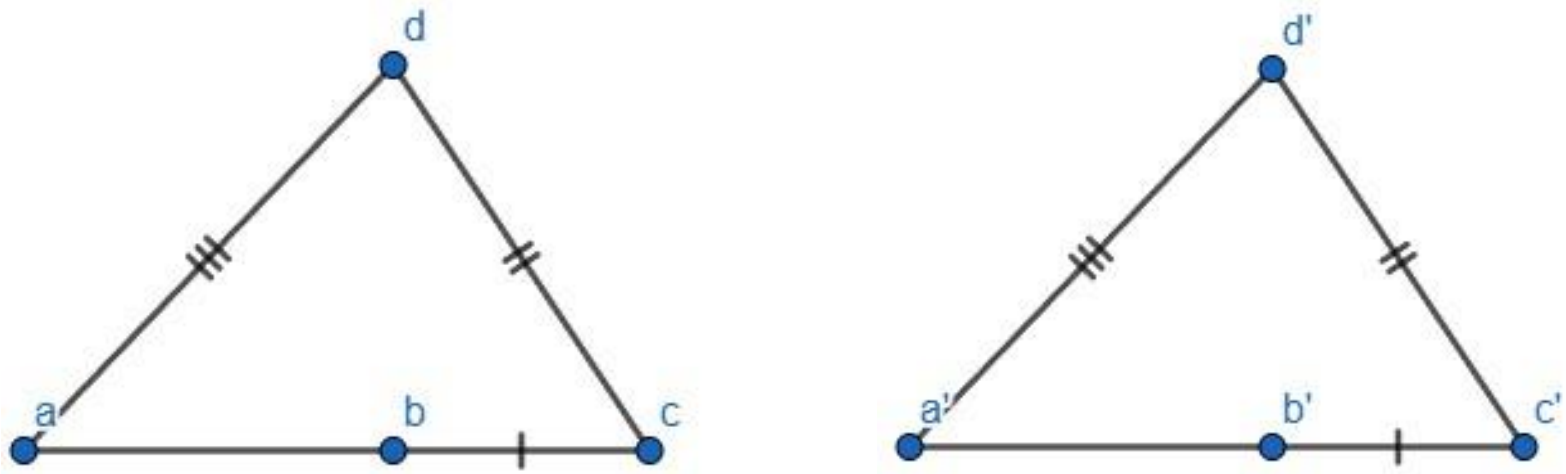
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Definition 4.1

- The points a, b, c, d, a′, b′, c′, d′ form a Inner-Five-Segment-Configuration
- $\mathrm{IFS}\begin{pmatrix} a & b & c & d \\ a' & b' & c' & d' \end{pmatrix} :\leftrightarrow \mathrm{Babc} \wedge \mathrm{Ba'b'c'} \wedge \mathrm{ac} \equiv a'c' \wedge \mathrm{bc} \equiv b'c' \wedge \mathrm{ad} \equiv a'd' \wedge \mathrm{cd} \equiv c'd'$

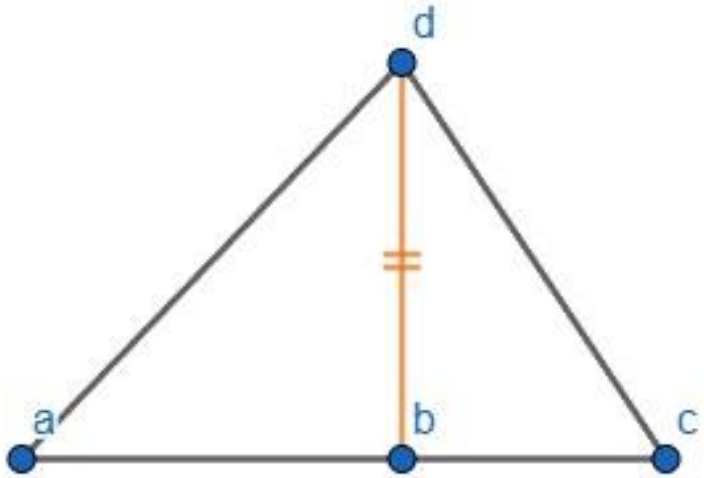
Definition 4.1

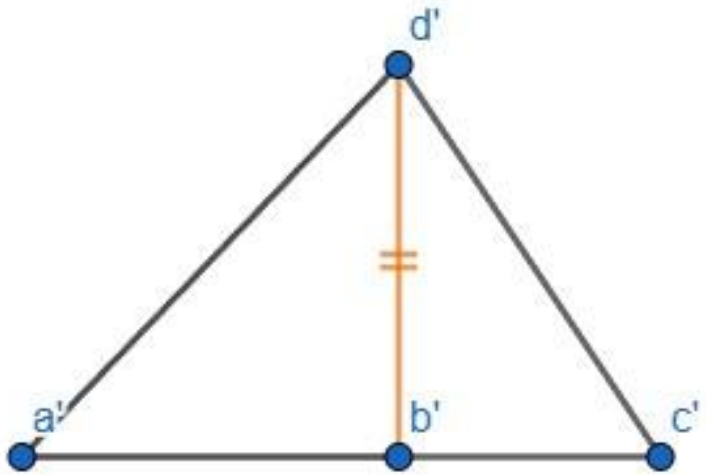
- The points a, b, c, d, a′, b′, c′, d′ form a Inner-Five-Segment-Configuration
- $\mathrm{IFS}\begin{pmatrix} a & b & c & d \\ a' & b' & c' & d' \end{pmatrix} :\leftrightarrow Babc \wedge Ba'b'c' \wedge ac \equiv a'c' \wedge bc \equiv b'c' \wedge ad \equiv a'd' \wedge cd \equiv c'd'$



Theorem 4.2

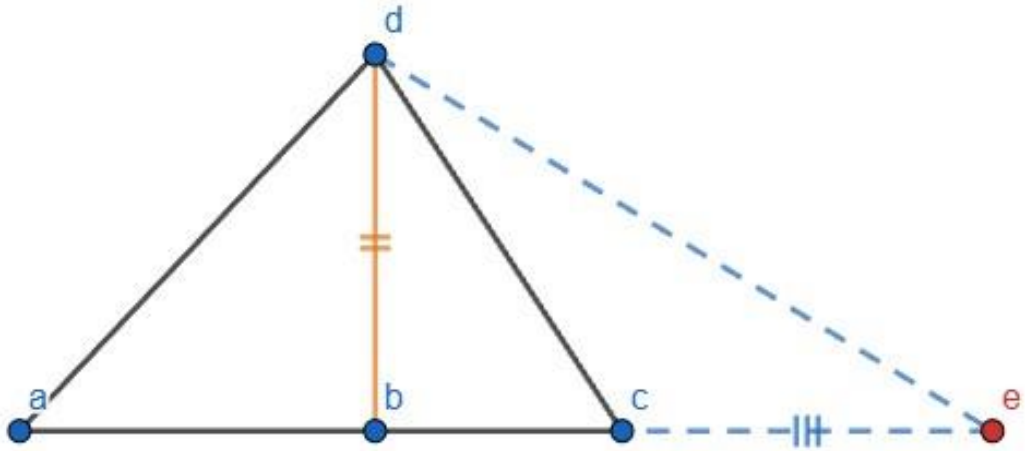
- IFS $\begin{pmatrix} a & b & c & d \\ a' & b' & c' & d' \end{pmatrix} \rightarrow \ bd \equiv \ b'd'$





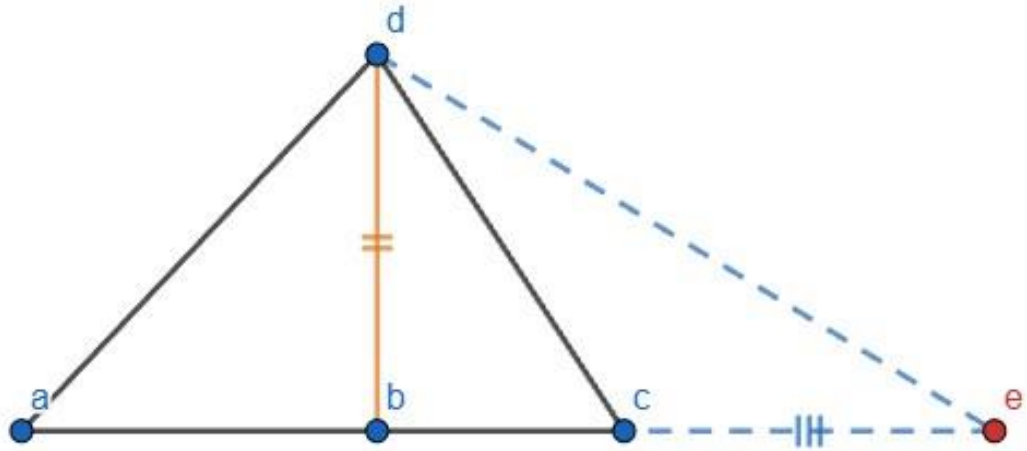
Case 1: a ≠ c

- Let e be as in 3.14, with Bace ∧ c ≠ e

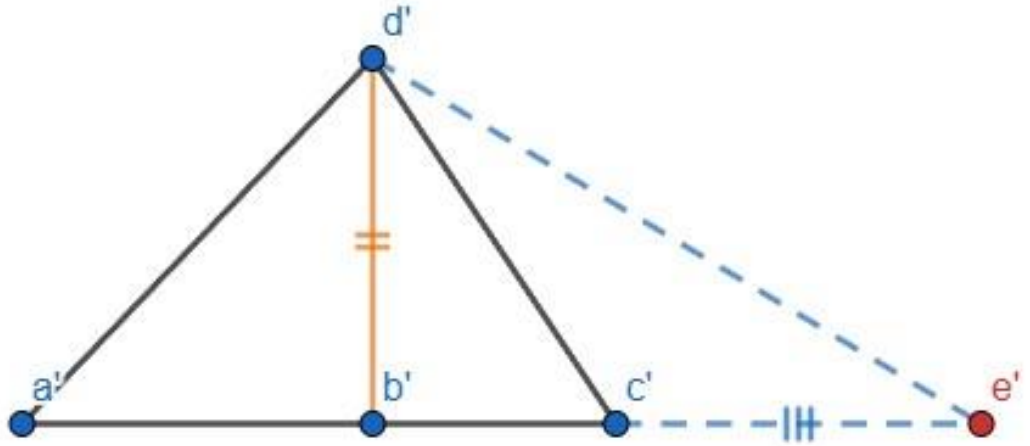


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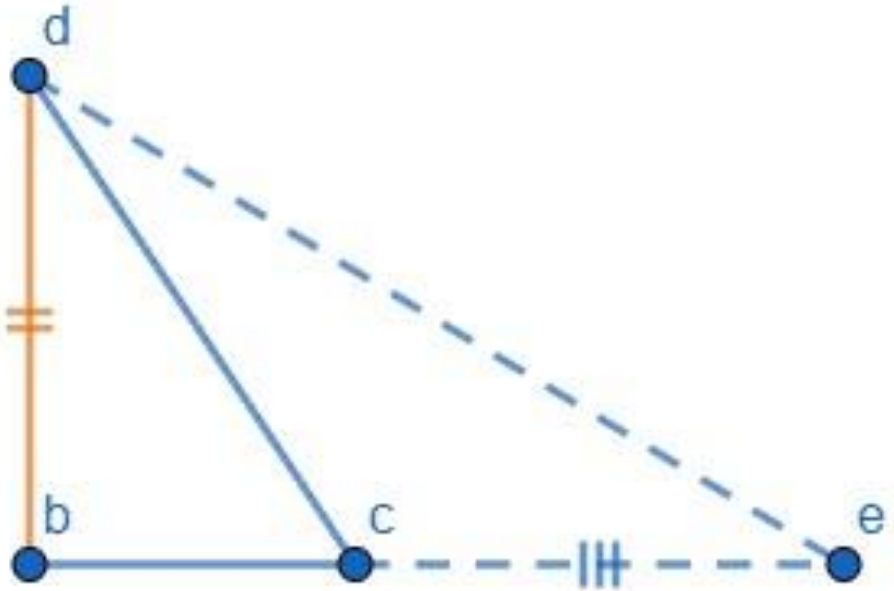
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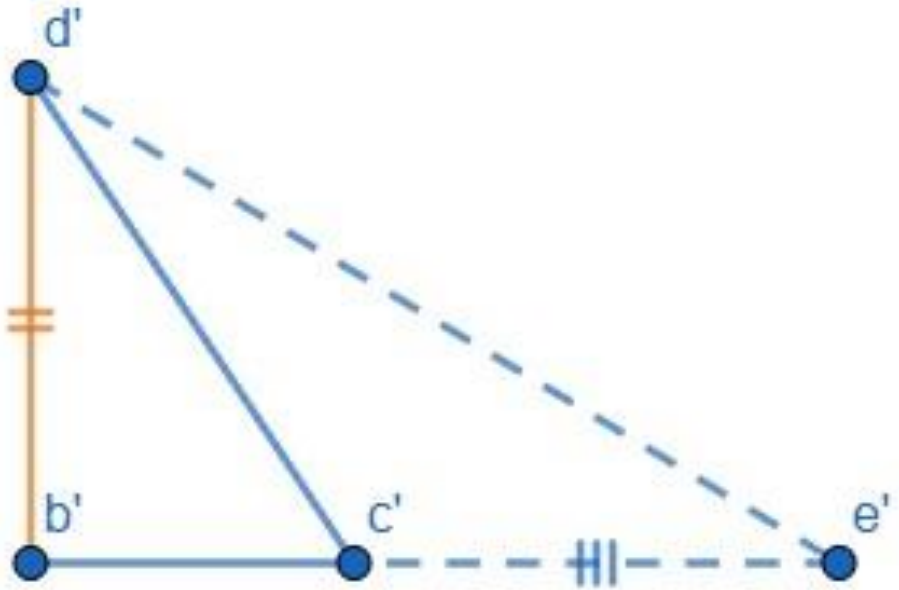


- Let e′ be as in A_4 , with Ba′c′e′ ∧ c′e′ ≡ ce

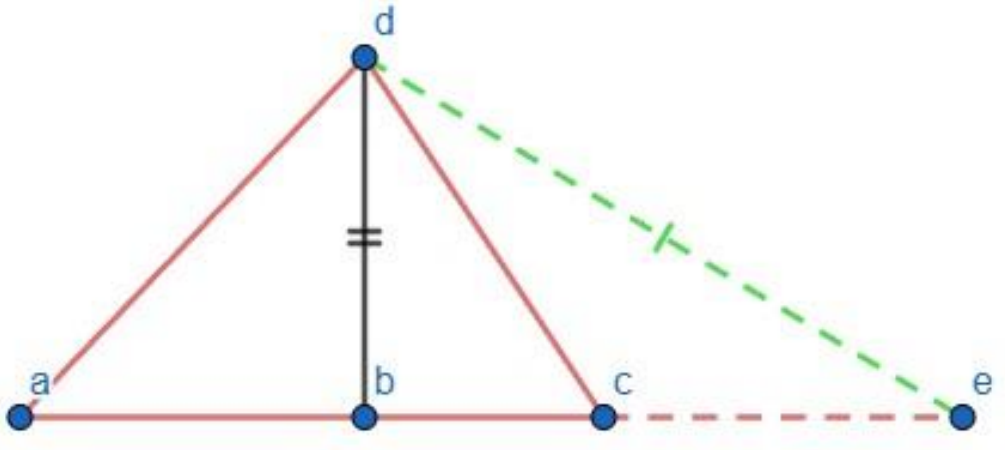


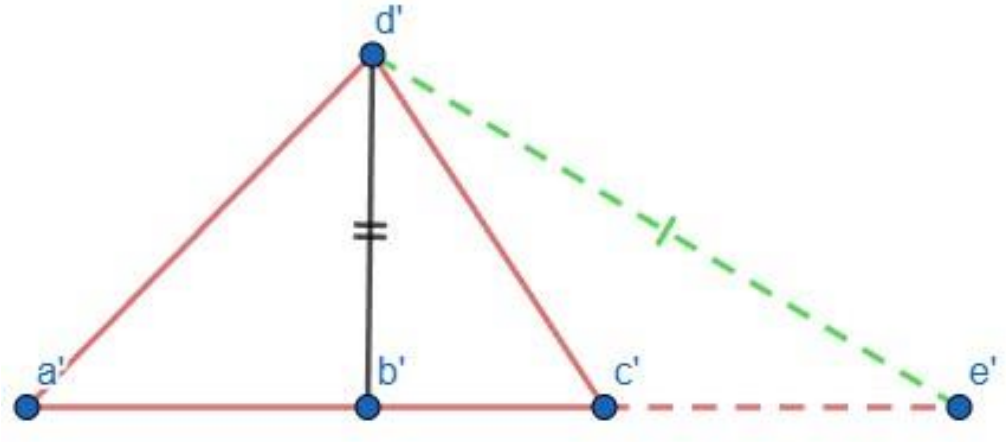
Case 1: $a \neq c$





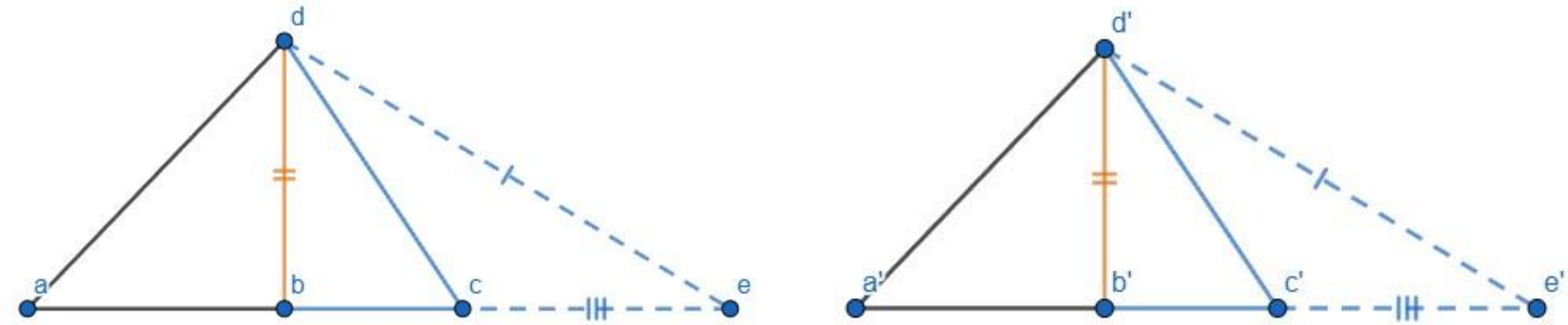
Case 1: a ≠ c





$$\mathrm{AFS}\begin{pmatrix} a & c & e & d \\ a' & c' & e' & d' \end{pmatrix} \xrightarrow{A_5} ed \equiv e'd'$$

Case 1: a ≠ c



Theorem 3.6 provides Babc ∧ Bace → Bbce

$$\mathrm{AFS}\begin{pmatrix} e & c & b & d \\ e' & c' & b' & d' \end{pmatrix} \xrightarrow{A_5} bd \equiv b'd'$$

Case 2: a = c

- $\mathrm{IFS}\begin{pmatrix} \mathrm{a} & b & c & d \\ \mathrm{a}' & b' & c' & d' \end{pmatrix} :\leftrightarrow \mathrm{Babc} \wedge \mathrm{Ba'b'c'} \wedge \mathrm{ac} \equiv \mathrm{a'}c' \wedge \mathrm{ad} \equiv \mathrm{a'd'} \wedge \mathrm{cd} \equiv \mathrm{c'd'}$

Case 2: $a = c$

- $\mathrm{IFS}\begin{pmatrix} a & b & c & d \\ a' & b' & c' & d' \end{pmatrix} :\leftrightarrow Babc \wedge Ba'b'c' \wedge ac \equiv a'c' \wedge ad \equiv a'd' \wedge cd \equiv c'd'$

- $a = c \overset{A_3}{\rightarrow} a' = c'$

Case 2: $a = c$

- $\mathrm{IFS}\begin{pmatrix} a & b & c & d \\ a' & b' & c' & d' \end{pmatrix} :\leftrightarrow Babc \wedge Ba'b'c' \wedge ac \equiv a'c' \wedge ad \equiv a'd' \wedge cd \equiv c'd'$

- $a = c \overset{A_3}{\rightarrow} a' = c'$
- $a = c \wedge a' = c' \overset{A_6}{\rightarrow} b = c \wedge b' = c'$

Case 2: $a = c$

- $\mathrm{IFS}\begin{pmatrix} a & b & c & d \\ a' & b' & c' & d' \end{pmatrix} :\leftrightarrow Babc \wedge Ba'b'c' \wedge ac \equiv a'c' \wedge ad \equiv a'd' \wedge cd \equiv c'd'$

- $a = c \xrightarrow{A_3} a' = c'$
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- $a = c \wedge a' = c' \xrightarrow{A_6} b = c \wedge b' = c' \rightarrow bd \equiv b'd'$

Definition 4.4

- The n-tuples $\langle a_1, \dots, a_n \rangle$ and $\langle b_1, \dots, b_n \rangle$ of points are considered conruent if :
- $(a_1 \dots a_n) \equiv (b_1 \dots b_n) :\leftrightarrow \bigwedge_{1 \leq i < j \leq n} a_i\, a_j = b_i b_j$.

Definition 4.10

- The points a, b, c are *collinear*:

  Colabc :↔ Babc ∨ Bbca ∨ Bcab

Definition 4.15

- The points a, b, c, d, a′, b′, c′, d′ form a Five-Segment-Configuration
- $\mathrm{FS}\begin{pmatrix} \mathrm{a} & b & c & d \\ \mathrm{a}' & b' & c' & d' \end{pmatrix} :\leftrightarrow \mathbf{Col}\mathrm{abc} \wedge (\mathrm{abc}) \equiv (\mathrm{a}'\mathrm{b}'\mathrm{c}') \wedge \mathrm{ad} \equiv \mathrm{a}'\mathrm{d}' \wedge \mathrm{bd} \equiv \mathrm{b}'\mathrm{d}'$

Theorem 4.16

- $\mathrm{FS}\begin{pmatrix} a & b & c & d \\ a' & b' & c' & d' \end{pmatrix} \wedge a \neq b \rightarrow cd \equiv c'd'$

Theorem 4.16

- $\mathrm{FS}\begin{pmatrix} a & b & c & d \\ a' & b' & c' & d' \end{pmatrix} \wedge a \neq b \rightarrow cd \equiv c'd'$

- Proof:

  This follows immediately from Axiom 5 and Theorem 4.2

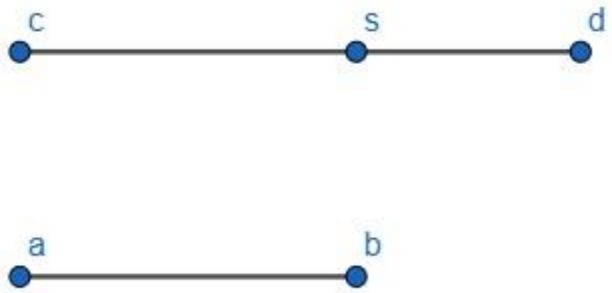
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Definition 5.4

- ab $\leq cd :\leftrightarrow \exists s[Bcsd \wedge ab \equiv cs]$

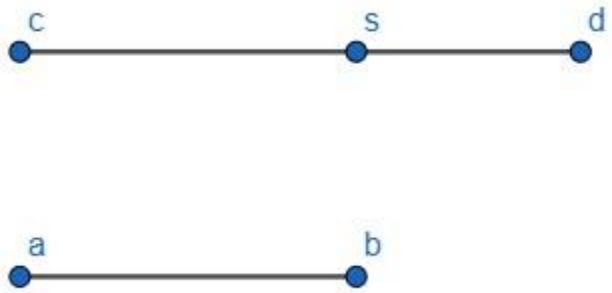
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Definition 5.4

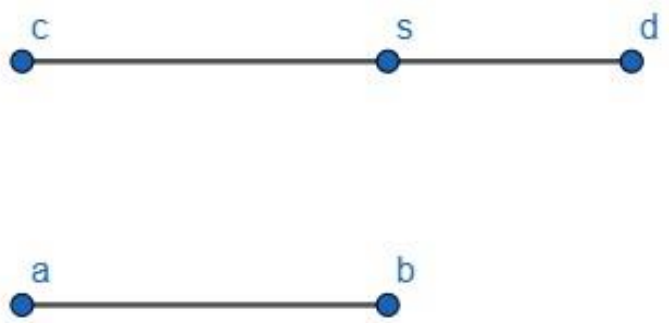
- $\mathrm{ab} \leq cd :\leftrightarrow \exists \mathrm{s}[B\mathrm{csd} \wedge \mathrm{ab} \equiv \mathrm{cs}]$



- $\mathrm{cd} \geq \mathrm{ab} :\leftrightarrow \mathrm{ab} \leq \mathrm{cd}$

Definition 5.14

- $ab < cd :\leftrightarrow ab \leq cd \wedge \neg ab \equiv cd$
- $cd > ab :\leftrightarrow ab < cd$

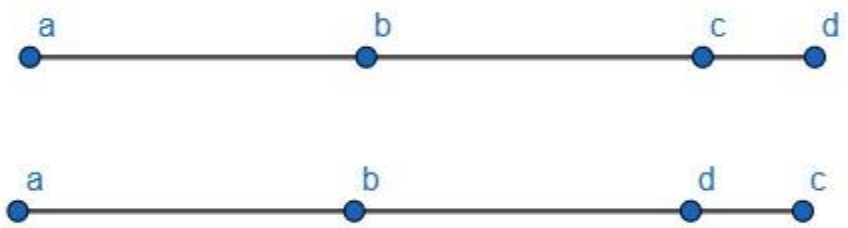


Theorem 5.1

- $\mathrm{a} \neq \mathrm{b} \wedge B\mathrm{abc} \wedge B\mathrm{abd} \rightarrow B\mathrm{acd} \vee B\mathrm{adc}$

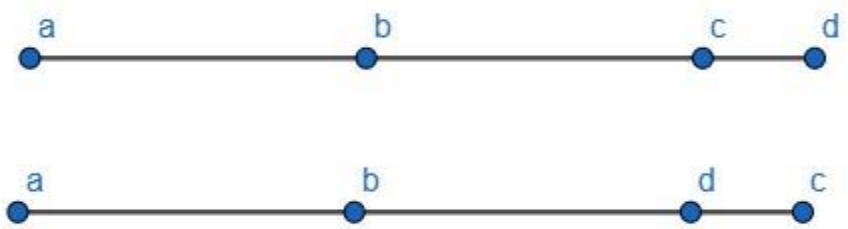
Theorem 5.1

- $a \neq b \wedge Babc \wedge Babd \rightarrow Bacd \vee Badc$



Theorem 5.1

- $a \neq b \wedge Babc \wedge Babd \rightarrow Bacd \vee Badc$



- $ac \leq ad \rightarrow Bacd$
- $ad \leq ac \rightarrow Badc$