Tarski’s Axioms

Chapter 1

outline

- points, betweenness and congruence
- axioms A1-A11
- cartesian space

points, betweenness, congruence

points betweenness congruence

only primitive object



points betweenness congruence

???

$$a \neq b \wedge Babc \wedge Ba'b'c' \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge ad \equiv a'd' \wedge bd \equiv b'd' \rightarrow cd \equiv c'd'$$

points betweenness congruence

only primitive object

characterized by lower-case letters



points betweenness congruence

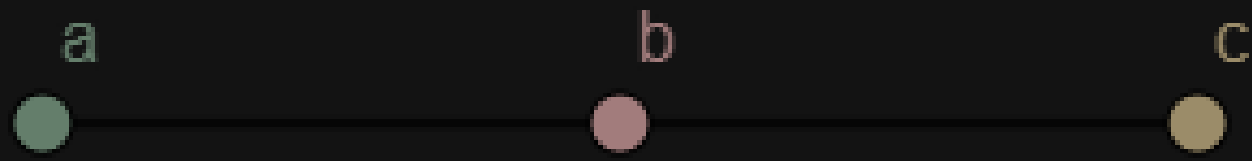
triadic relation

$$Babc$$

points betweenness congruence

triadic relation

$Babc$



b *lies between* ***a*** *and* ***c***

points betweenness congruence

↔ “equidistance”

tetradic relation

$$ab \equiv cd$$

points betweenness congruence

↔ “equidistance”

tetradic relation

$$ab \equiv cd$$

ab** is congruent to **cd

the distance between **a** and **b** is equal to the distance between **c** and **d**

Axioms

A1

Reflexivity of congruence

$$ab \equiv ba$$

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$$\forall a, b \in S\colon (ab \equiv ba)$$

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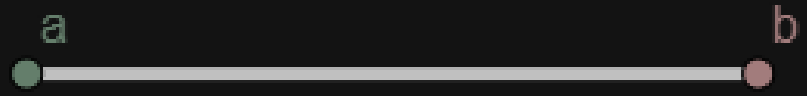
ab is congruent to ba

A1

Reflexivity of congruence

$$ab \equiv ba$$

ab *is congruent to* ***ba***



A2 Transitivity of congruence

$$ab \equiv pq \land ab \equiv rs \rightarrow pq \equiv rs$$

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$$ab \equiv pq \land ab \equiv rs \rightarrow pq \equiv rs$$

*If **ab** is congruent to **pq** and **ab** is congruent to **rs**, then **pq** must be congruent to **rs***

A3 Identity of congruence

$$ab \equiv cc \rightarrow a = b$$

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$$ab \equiv cc \rightarrow a = b$$

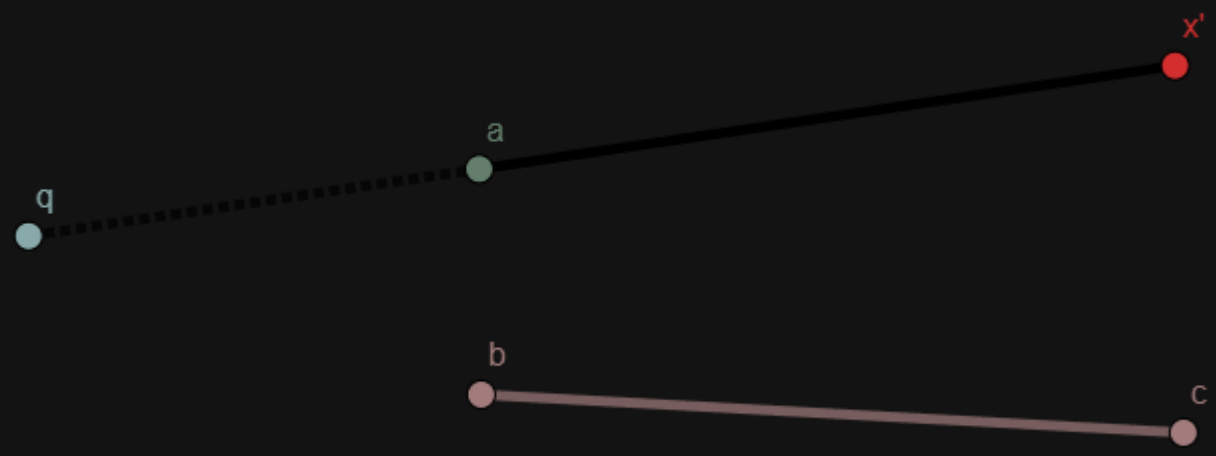
*If **ab** is congruent to **cc**, then **a** must be equal to **b***

A4 Segment construction

$$\exists x(Bqax \wedge ax \equiv bc)$$

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$$\exists x(Bqax \wedge ax \equiv bc)$$



A5

Five segment axiom

$$a \neq b \wedge Babc \wedge Ba'b'c' \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge ad \equiv a'd' \wedge bd \equiv b'd' \rightarrow cd \equiv c'd'$$

A5

Five segment axiom

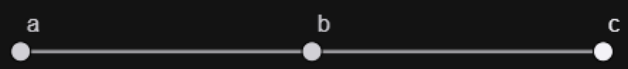
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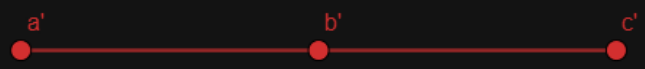


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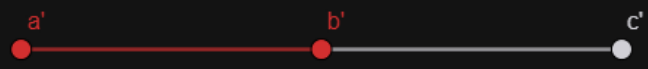


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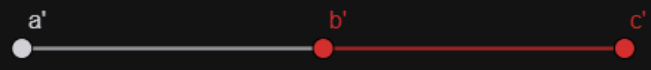


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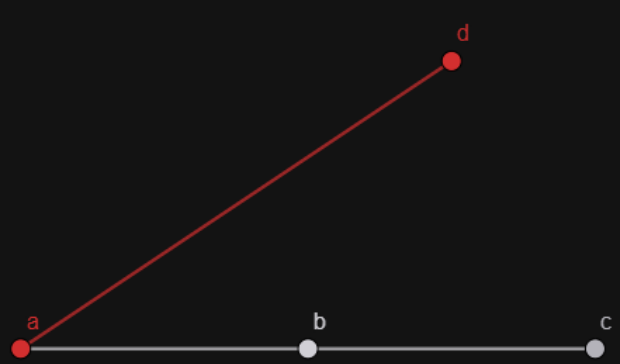


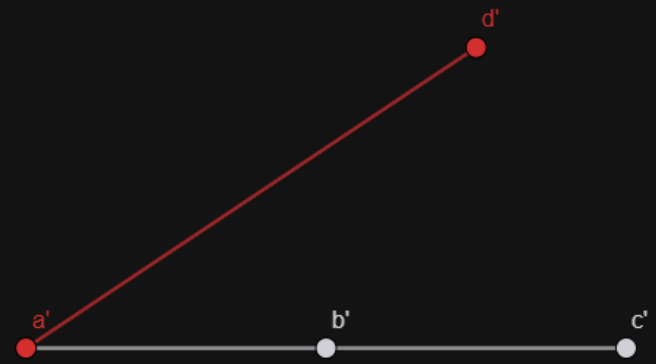


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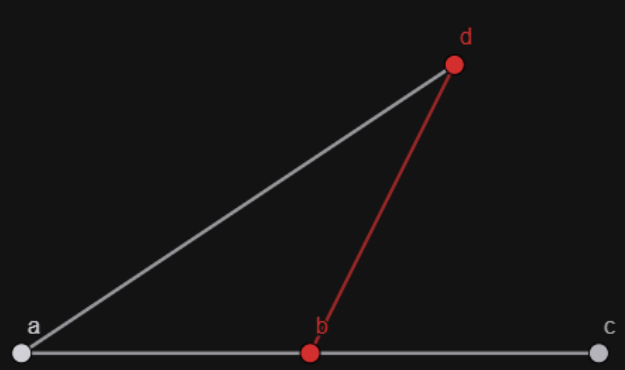


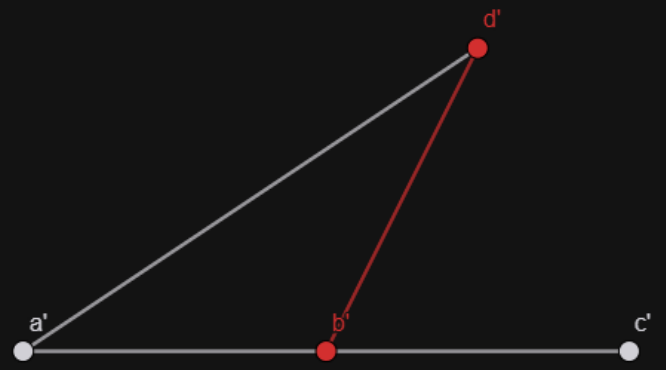


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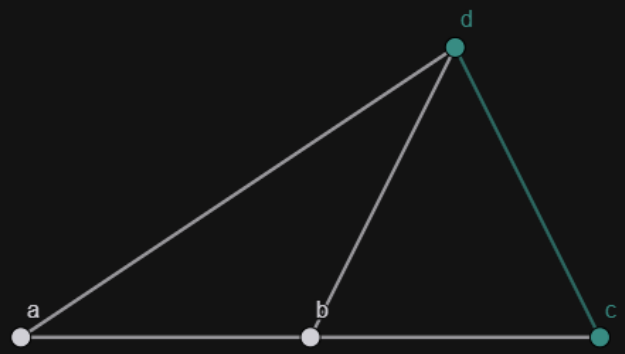


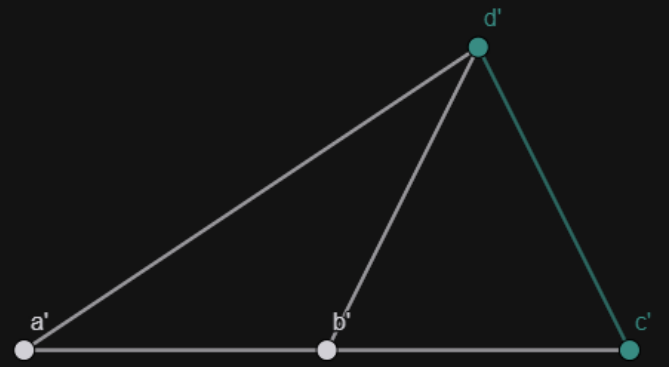


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A6 Identity of betweenness

$$Baba \to a = b$$

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$$Baba \rightarrow a = b$$

*If **b** is between **a** and **a**, then **a** must be equal to **b**.*

A7

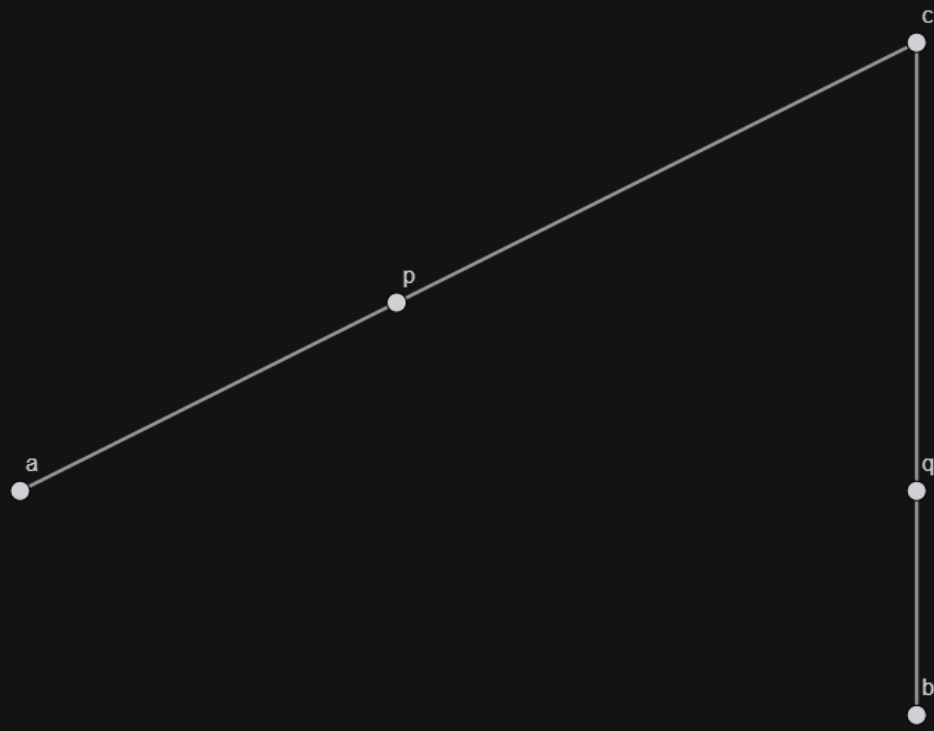
Axiom of Pasch

$$Bapc \wedge Bbqc \rightarrow \exists x(Bpxb \wedge Bqxa)$$

A7

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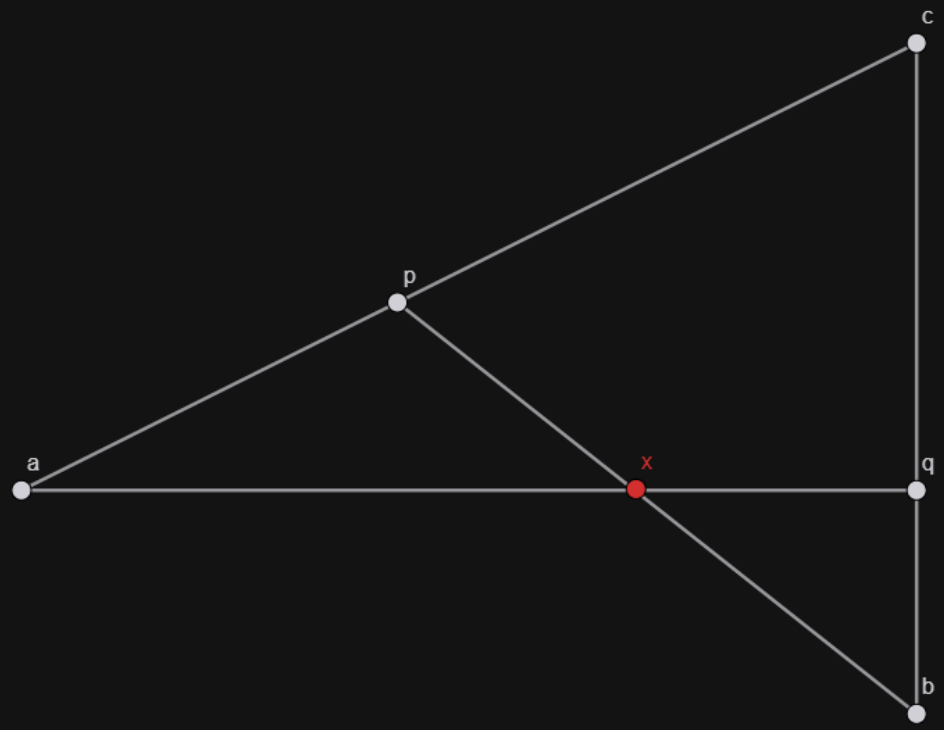
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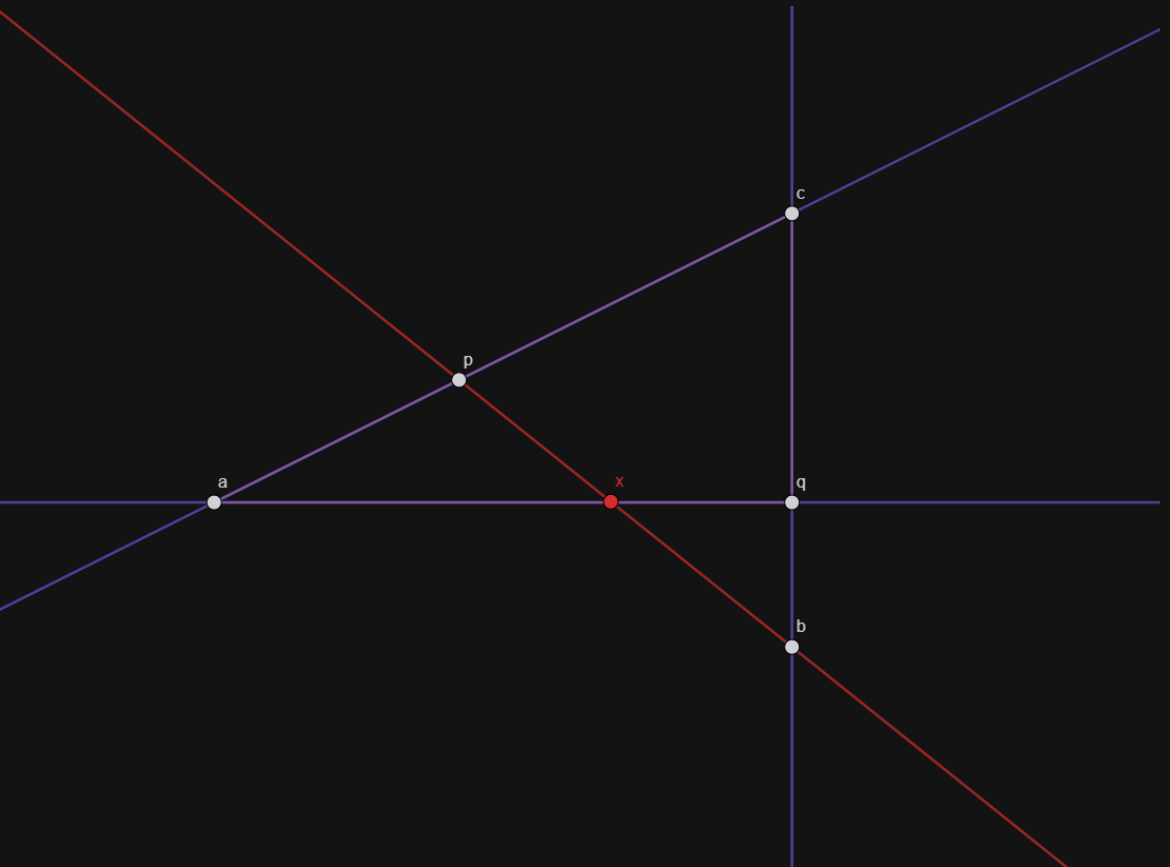
$$Bapc \wedge Bbqc \rightarrow \exists x(Bpxb \wedge Bqxa)$$

*If a line intersects one side of a triangle **internally**, then it intersects precisely one other side **internally** and the third side **externally**, if it does not pass through any corner.*

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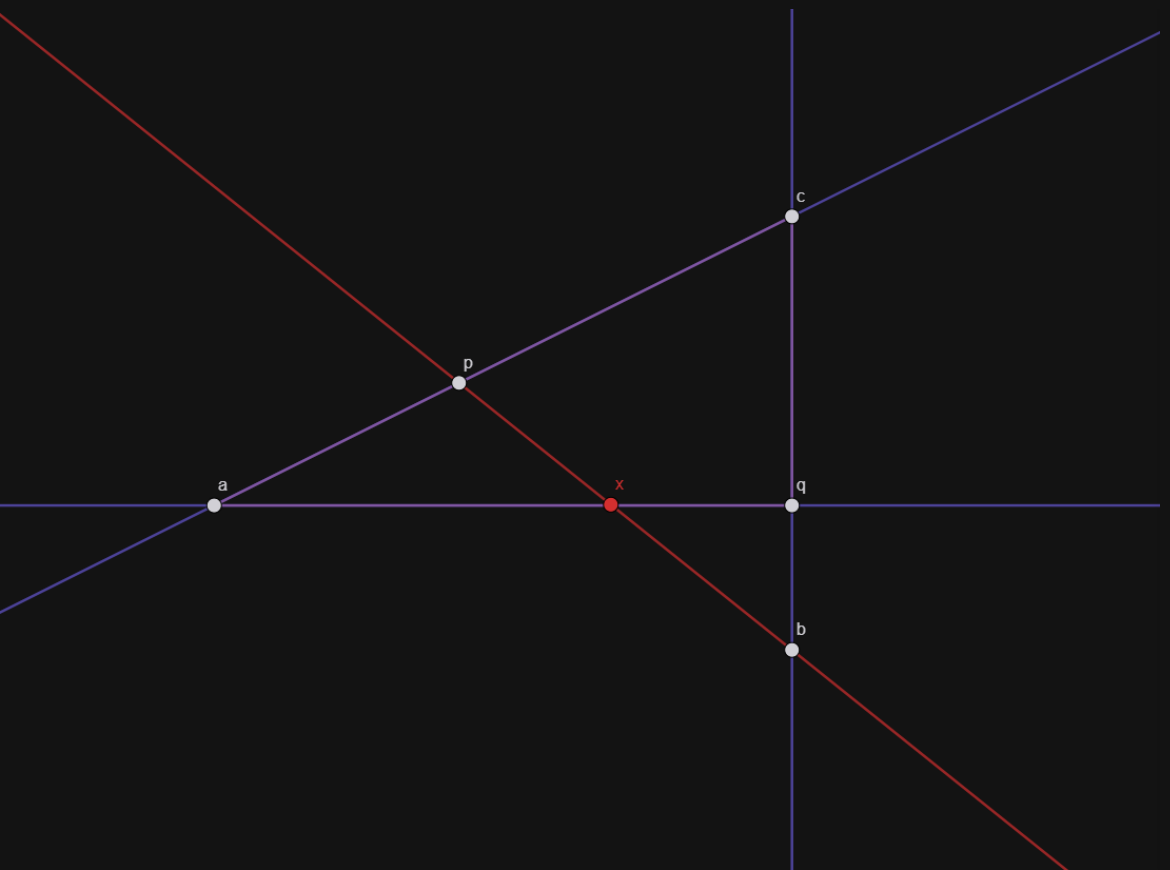


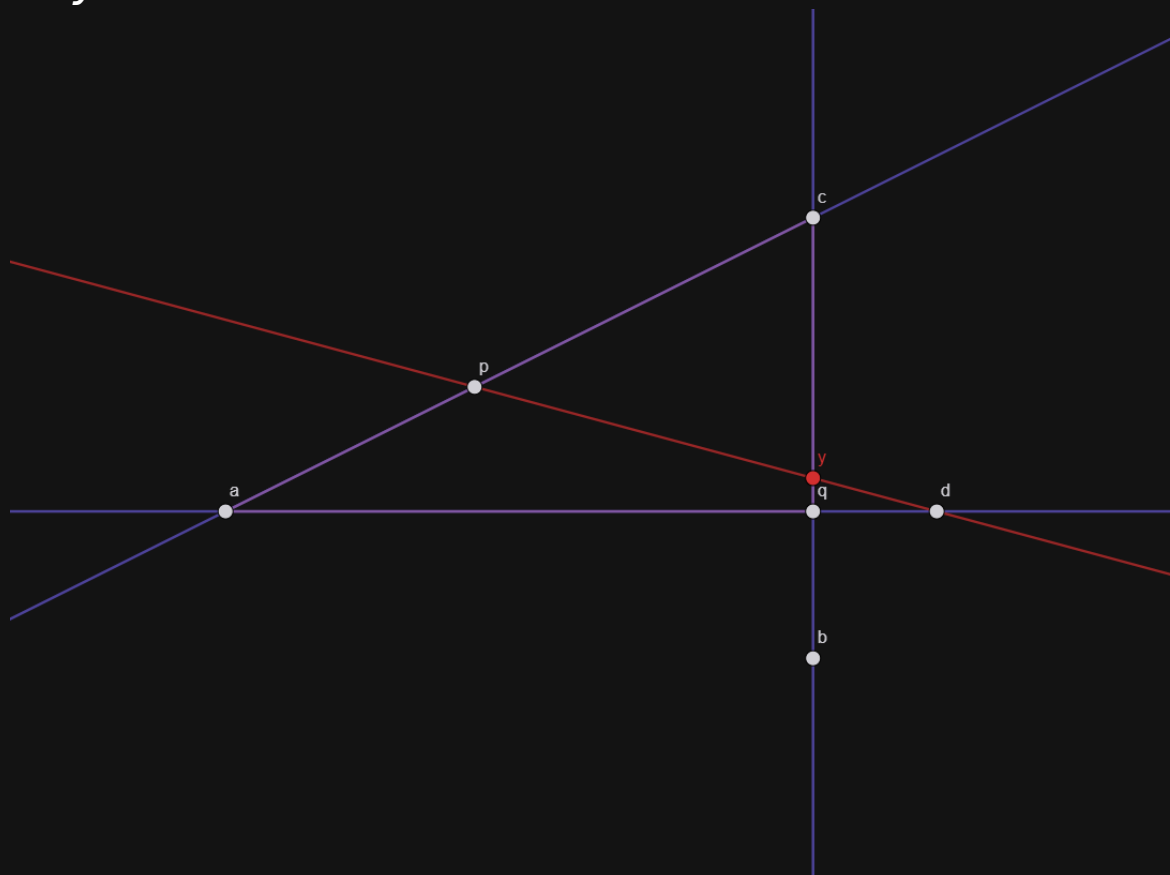
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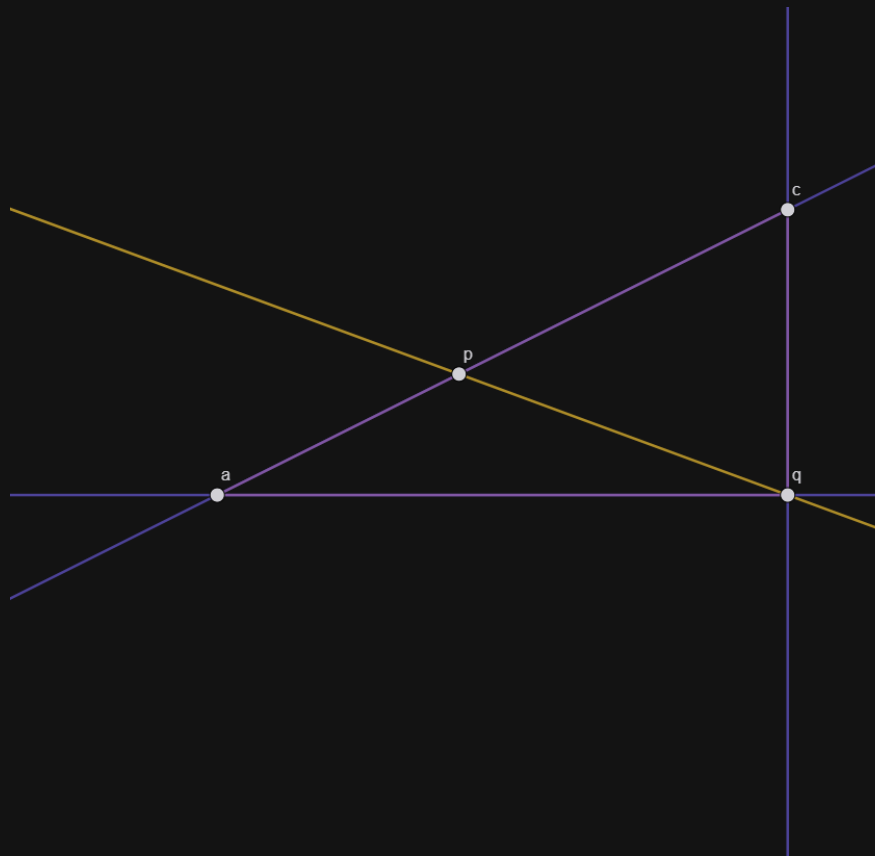




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A8 Lower dimension axiom

$$\exists a \exists b \exists c (\neg Babc \wedge \neg Bbca \wedge \neg Bcab)$$

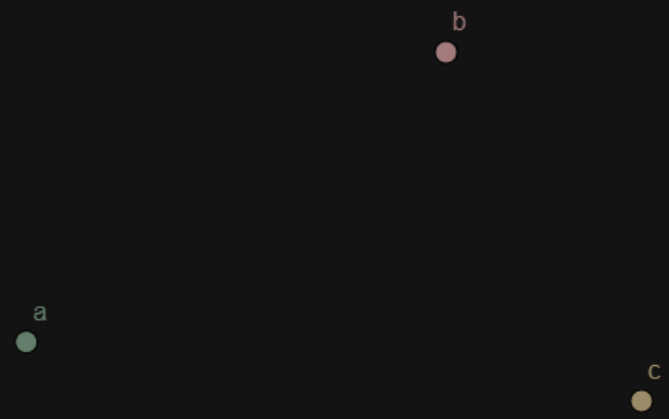
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*There exist points **a b c** that are not “colinear”*

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A9

Upper dimension axiom

$$p \neq q \wedge ap \equiv aq \wedge bp \equiv bq \wedge cp \equiv cq \rightarrow Babc \vee Bbca \vee Bcab$$





A9 Upper dimension axiom

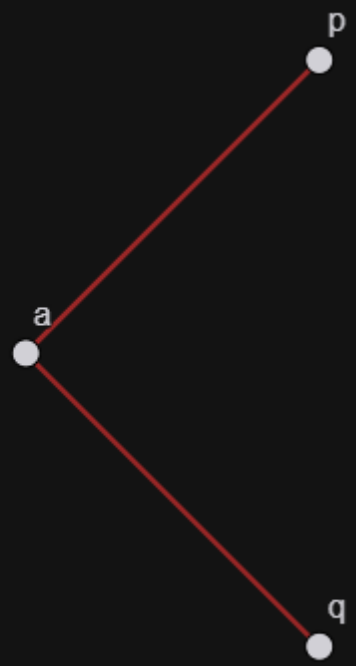
$$p \neq q \wedge apq \equiv aqp \wedge bpq \equiv bqp \wedge cpq \equiv cqp \rightarrow \underbrace{Babc \vee Bbca \vee Bcab}_{\text{“all 3 points lie on a line”}}$$





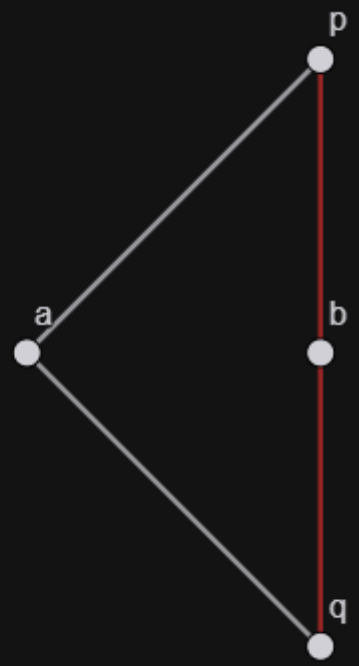
A9 Upper dimension axiom

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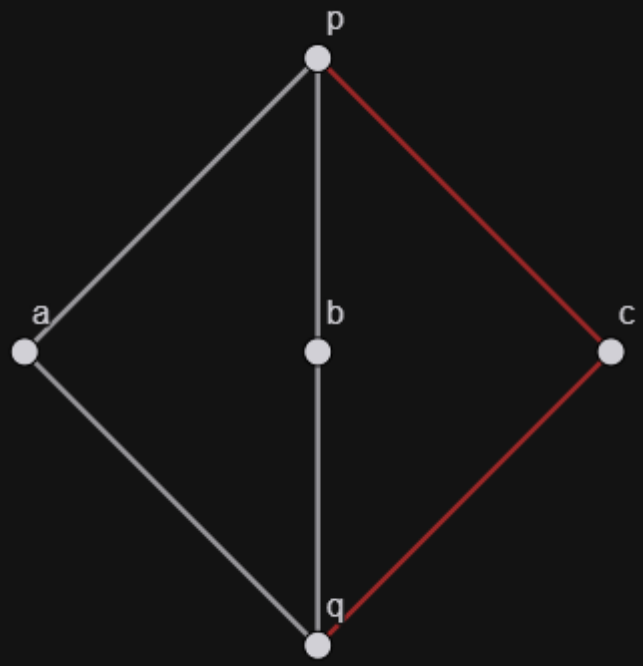
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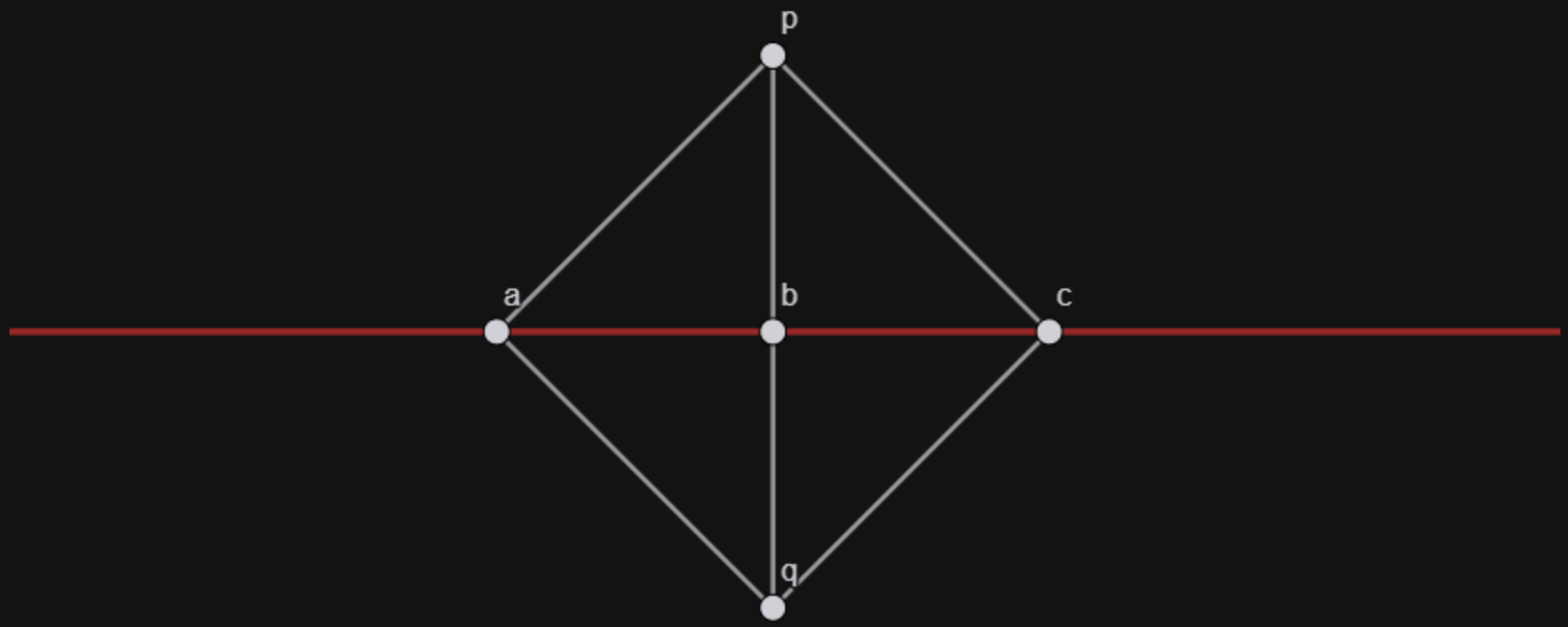
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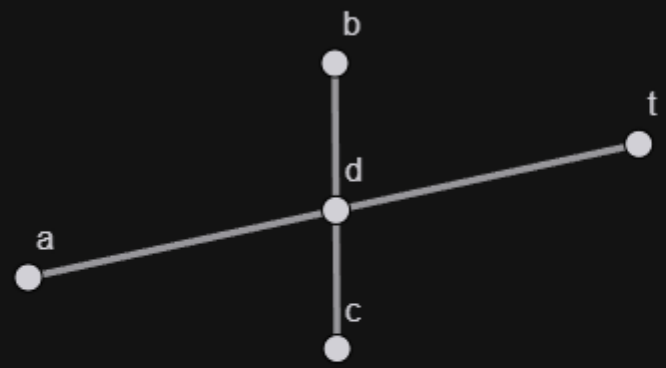


A10 Axiom of Euclid

$$Badt \wedge Bbdc \wedge a \neq b \rightarrow \exists x \exists y (Babx \wedge Bacy \wedge Bxty)$$

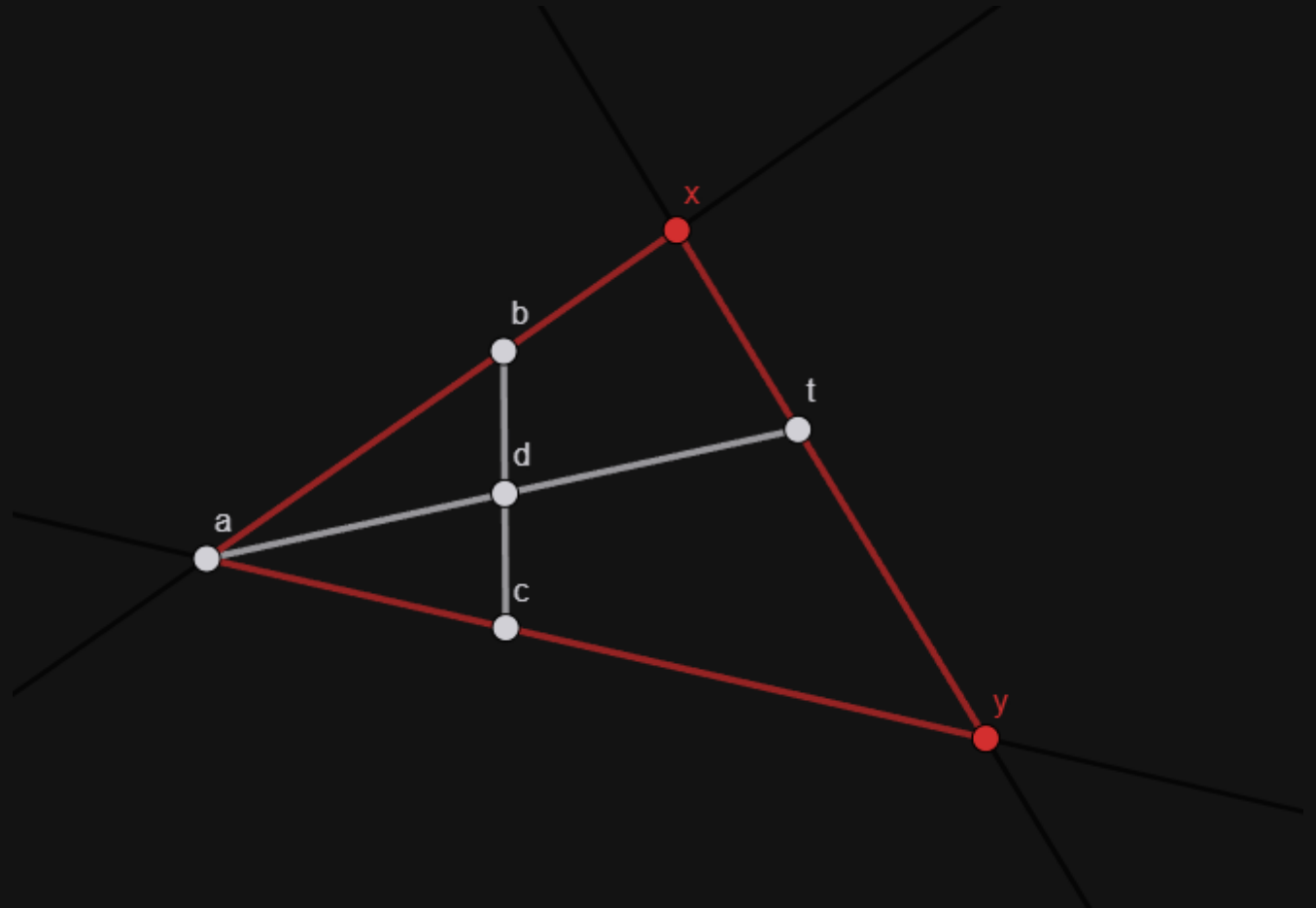
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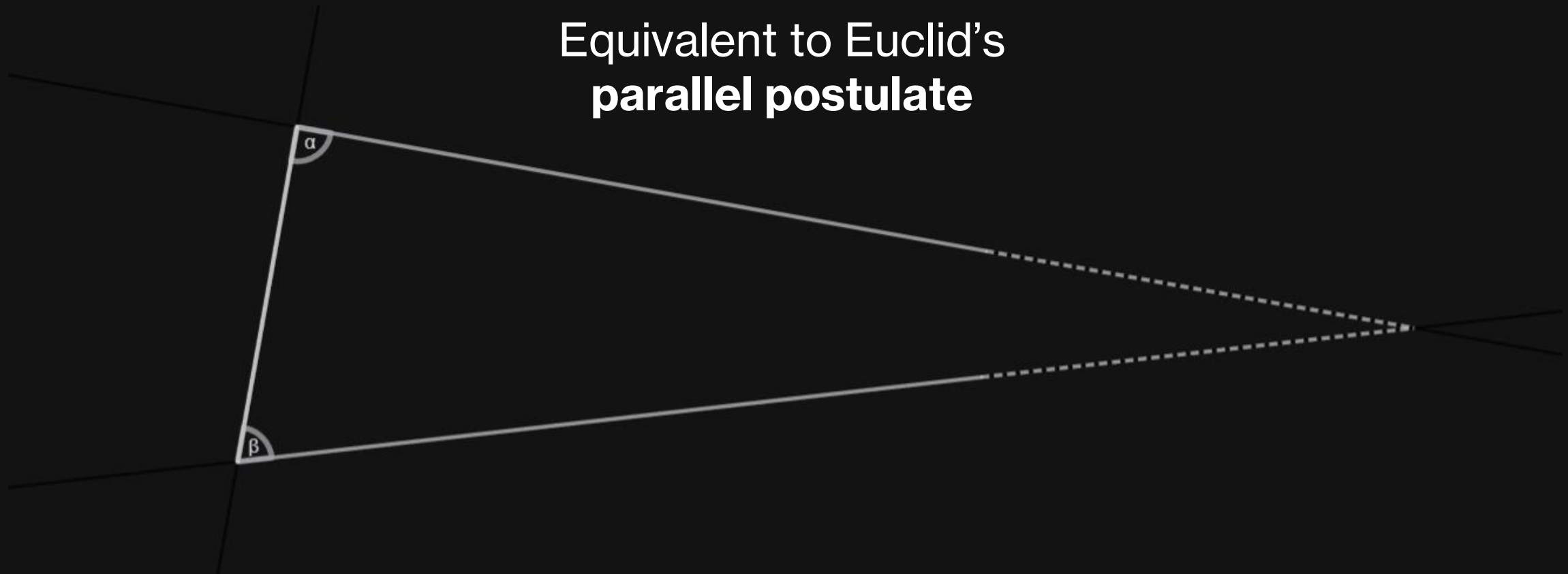
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Equivalent to Euclid's
parallel postulate

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A11 Axiom of continuity

$$\forall X \forall Y \{\exists a \forall x \forall y [x \in X \wedge y \in Y \rightarrow Baxy] \rightarrow \exists b \forall x \forall y [x \in X \wedge y \in Y \rightarrow Bxby]\}$$

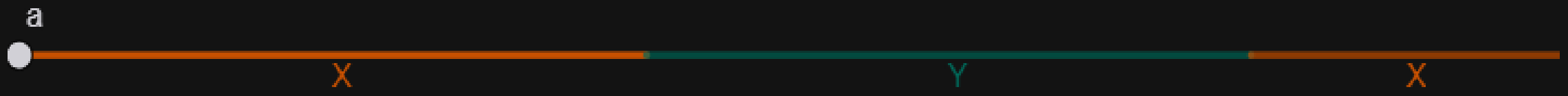
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A11' Axiom schema of continuity

$$\exists a \forall x \forall y [\alpha(x) \wedge \beta(y) \rightarrow Baxy] \rightarrow \exists b \forall x \forall y [\alpha(x) \wedge \beta(y) \rightarrow Bxby]$$

formulas using B and $\equiv$ $\begin{cases} \alpha(x) \leftrightarrow x \in X \\ \beta(y) \leftrightarrow y \in Y \end{cases}$

definition 1.4:
cartesian space

ordered field $\mathcal{F} = < F, +, \cdot, \leq >$

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n-dimensional cartesian space $\mathcal{L}_n(\mathcal{F}) = < F^n, D^n_{\mathcal{F}}, B^n_{\mathcal{F}} >$

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cartesian space

ordered field $\mathscr{F} = < F, +, \cdot, \leq >$

$a = (a_1, a_2, \ldots, a_n)$

n-dimensional cartesian space $\mathscr{L}_n(\mathscr{F}) = < F^n, D^n_{\mathscr{F}}, B^n_{\mathscr{F}} >$

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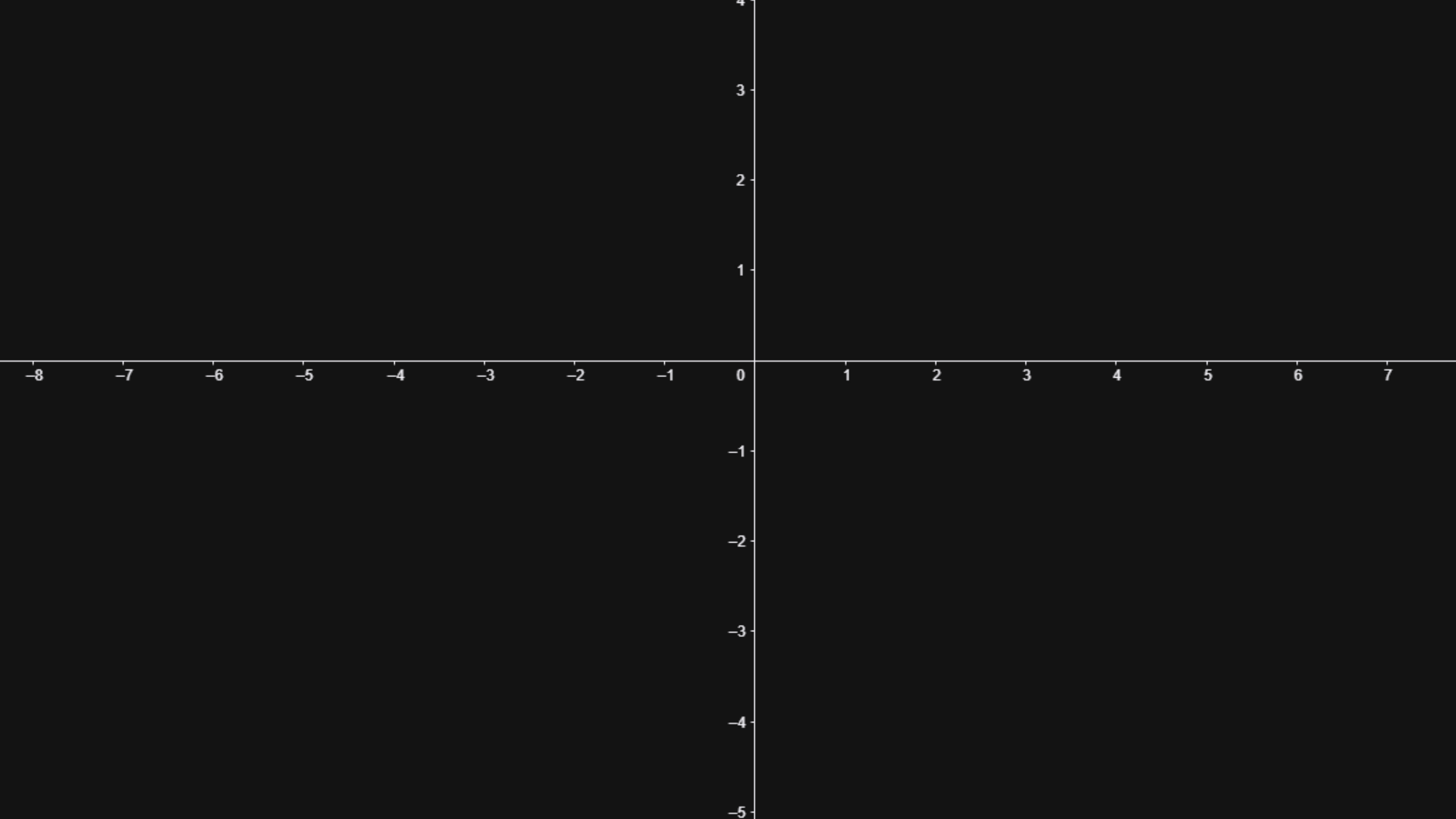
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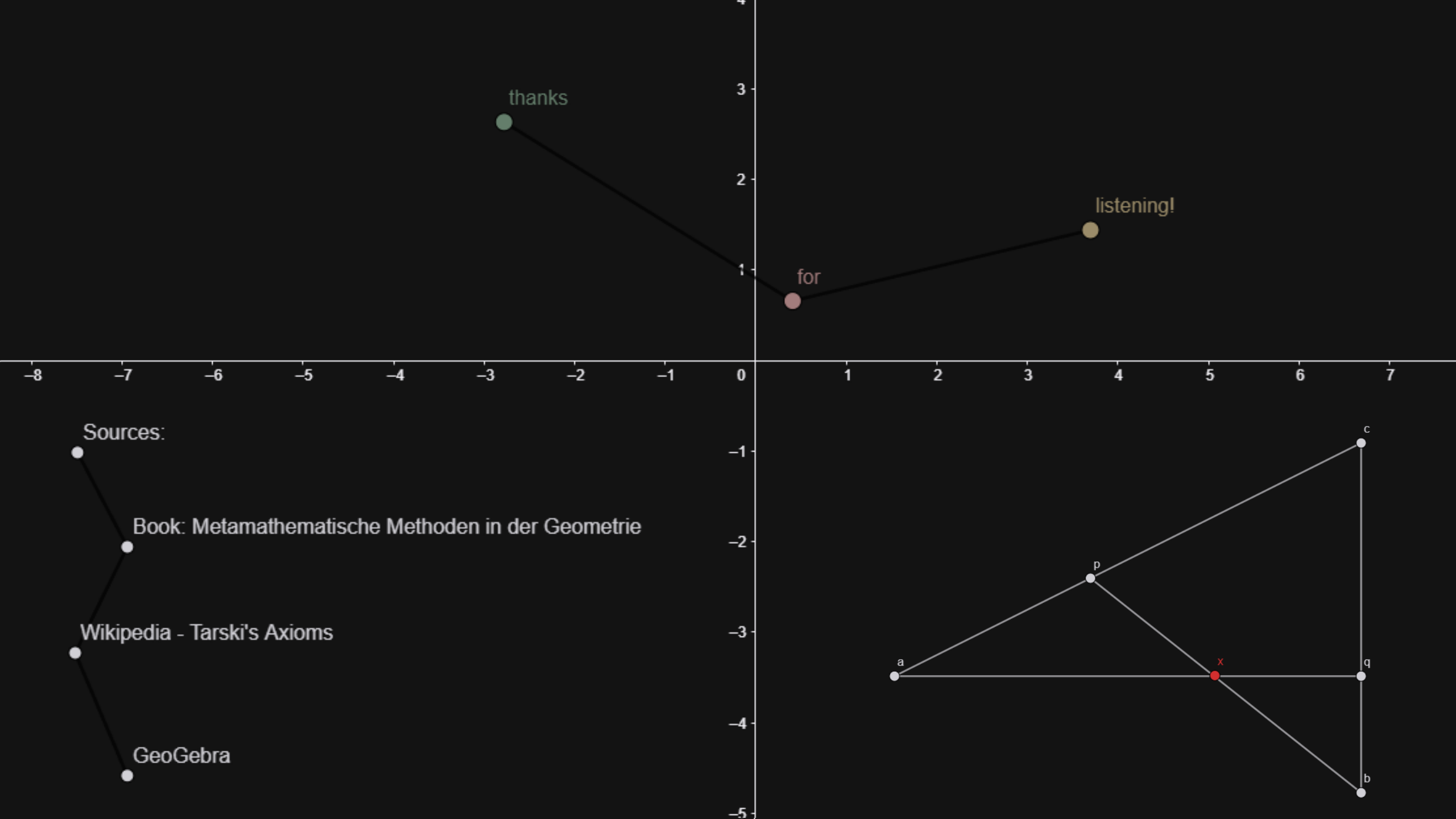
$\mathscr{L}_n(\mathscr{F})$ fulfills all axioms if:

$n = 2$

$\mathscr{F} = (\mathbb{R}, +, \cdot, \leq)$

Further reading: p.16-17


4
3
2
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−8
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−6
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−1
0
1
2
3
4
5
6
7
−1
−2
−3
−4
−5


thanks
for
listening!
Sources:
Book: Metamathematische Methoden in der Geometrie
Wikipedia - Tarski's Axioms
GeoGebra
a
p
c
x
q
b