Half-planes, Planes, Subspaces

Handout – Chapter 9: from points and lines to planes and higher dimensions

Notation. $\mathrm{Ln}\,A$: A is a line; $L(pq)$: the line through $p \neq q$; $\mathrm{B}\,atb$: t lies between a and b; $\mathrm{Col}\,abc$: a, b, c are collinear; $a \simeq_r b$: a and b lie on the same ray starting at r.

Guiding idea. A line divides the plane into two *sides*. We make this precise using only *betweenness*: two points are on *different* sides if the segment between them crosses the line, and on the *same* side if they are both on the other side of some third point. The sides are then obtained as *equivalence classes*, and the whole construction is repeated in every dimension.

1 Opposite points

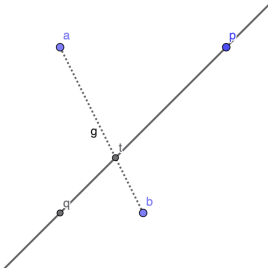
Intuition: a and b lie on *opposite* sides of the line A if neither lies on A and the segment from a to b meets A.

9.1 Definition (opposite) a lies *opposite* to b with respect to A $(= L(pq))$:

$$a \,|_A\, b \;:\Longleftrightarrow\; \mathrm{Ln}\,A \wedge a \notin A \wedge b \notin A \wedge \exists t\,[\,t \in A \wedge \mathrm{B}\,atb\,].$$

9.5 Theorem $a \,|_A\, c \wedge r \in A \;\rightarrow\; \forall b\,[\,a \simeq_r b \rightarrow b \,|_A\, c\,]$

Moving a along a ray that starts on A never changes its side.



$a \,|_A\, b$: the segment ab meets $A = L(pq)$ in t.

2 Equivalent points

Intuition: a and b are *equivalent* ("on the same side") with respect to A if there is a witness point c lying opposite to both of them. Equivalently: a, b define the same half-plane with A as its boundary.

9.7 Definition (equivalent)

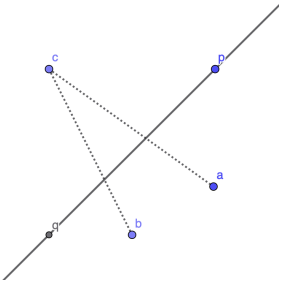
$$a \simeq_A b \;:\Longleftrightarrow\; \exists c\,[\,a \,|_A\, c \wedge b \,|_A\, c\,].$$

Theorems 9.8 – 9.10

9.8 $a \,|_A\, c \rightarrow (\,b \,|_A\, c \leftrightarrow a \simeq_A b\,)$ — *the witness c can be any opposite point*

9.9 $a \,|_A\, b \rightarrow \neg\, a \simeq_A b$ — *opposite points are never equivalent*

9.10 $\mathrm{Ln}\,A \wedge a \notin A \rightarrow \exists c:\; a \,|_A\, c$ — *every point has an opposite point*



$a \simeq_A b$: both are opposite to the witness c.

3 Half-planes as equivalence classes

9.11 – 9.13 Theorems (from 9.7, 9.8, 9.10) For every line A, $\simeq_A$ is an **equivalence relation** on the set of points not on A:

Reflexivity: $\mathrm{Ln}\,A \wedge a \notin A \rightarrow a \simeq_A a$

Symmetry: $a \simeq_A b \rightarrow b \simeq_A a$

Transitivity: $a \simeq_A b \wedge b \simeq_A c \rightarrow a \simeq_A c$

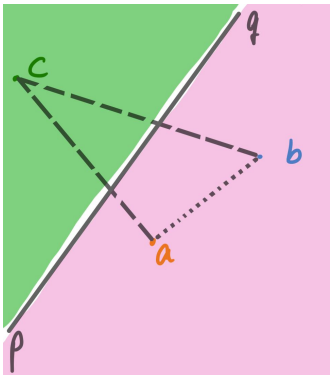
Why exactly two classes? By 9.10 every point a has an opposite point c; by 9.8 every point not on A is either equivalent to a or opposite to a (then equivalent to c); by 9.9 these cases exclude each other.

9.14 Definition (half-plane) The two equivalence classes of $\simeq_A$ are the *half-planes starting from A*. The class of a is the half-plane originating at A through a:

$$\forall\,\mathrm{Ln}\,A \wedge a \notin A:\quad \mathrm{Hp}(A,a) := \{\,x \mid x \simeq_A a\,\}.$$

9.15 $\mathrm{Hp}(A,a)$ and $\mathrm{Hp}(A,b)$ are called *opposite* if $a \,|_A\, b$ (independent of the choice of a, b).

9.17 Theorem (convexity) Half-planes are *convex*: $a \simeq_A b \wedge \mathrm{B}\,acb \;\rightarrow\; c \simeq_A a$.

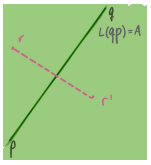


The line $A = L(pq)$ splits the points into two classes: green $= \mathrm{Hp}(A,c)$, pink $= \mathrm{Hp}(A,a) = \mathrm{Hp}(A,b)$. Dashed: opposite pairs; dotted: equivalent pair.

4 Planes

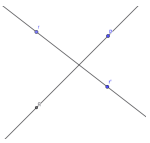
Intuition: a plane is a line together with both of its sides: glue the two opposite half-planes to their common boundary line.

9.20 Definition (plane) For $\mathrm{Ln}\, A, r \notin A$ and $r \mid_A r'$ the plane through A and r is
$$\mathrm{Pl}(A,r) := \mathrm{Hp}(A,r) \cup A \cup \mathrm{Hp}(A,r').$$
With $\mathrm{Pl}(pqr) := \mathrm{Pl}(L(pq),r)$, a set of points E is a *plane*:
$$\mathrm{Pl}E \iff \exists p\, \exists q\, \exists r :\ \neg\mathrm{Col}\, pqr \wedge E = \mathrm{Pl}(pqr).$$



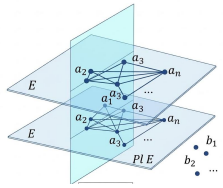
$\mathrm{Pl}(A,r), r \mid_A r', A = L(qp)$.

9.23 Definition For intersecting lines A, A' with $r \in A', r \notin A$: $\mathrm{Pl}(A,A') := \mathrm{Pl}(A,r)$ (independent of r by 9.21, 9.22).


Plane of two intersecting lines (9.23).

9.25 Theorem A line lies completely in a plane if it passes through two distinct points of that plane:
$$\mathrm{Pl}E \wedge a \in E \wedge b \in E \wedge a \neq b \ \to\ L(ab) \subseteq E \wedge \exists c :\ E = \mathrm{Pl}(abc).$$

9.32 Definition (coplanar) Points / lines are *coplanar* if they lie in a common plane:
$$\mathrm{cop}\, a_1 \dots a_n :\Leftrightarrow \exists E :\ \mathrm{Pl}E \wedge \textstyle\bigwedge_{\nu=1}^{n} a_\nu \in E,$$
$$\mathrm{cop}\, A_1 \dots A_n :\Leftrightarrow \exists E :\ \mathrm{Pl}E \wedge \textstyle\bigwedge_{\nu=1}^{n} (\mathrm{Ln}\, A_\nu \wedge A_\nu \subseteq E).$$



Coplanar points $a_1, \dots, a_n$ in E; $b_1, b_2, \dots$ outside.

9.33 Theorem $\mathrm{cop}\, abcd \ \Leftrightarrow\ \exists x\, [\, (\mathrm{Col}\, abx \wedge \mathrm{Col}\, cdx) \vee (\mathrm{Col}\, acx \wedge \mathrm{Col}\, bdx) \vee (\mathrm{Col}\, adx \wedge \mathrm{Col}\, bcx)\,]$ *– four points are coplanar iff, split into two pairs, the two connecting lines share a point x.*

9.34 Theorem (plane as closure) If $\neg\mathrm{Col}\, abc$, then $\mathrm{Pl}(abc)$ is the *smallest* set E with (1) $a, b, c \in E$ and (2) E closed under connecting lines. $\Rightarrow$ every point $d \in \mathrm{Pl}(abc)$ can be constructed with two connecting lines.

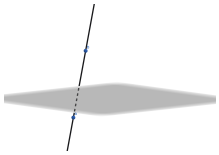
5 Subspaces: transfer to higher dimensions

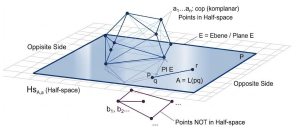
Idea: The construction "boundary $\to$ opposite / equivalent points $\to$ two sides (equivalence classes) $\to$ glue together" never used that A is a *line*. Replacing the line by an n-dimensional subspace A ($\mathrm{Sub}_n A$) yields an $(n+1)$-dimensional subspace: $\dim(\text{point}) = 0,\ \dim(\text{line}) = 1,\ \dim(\text{plane}) = 2,\ \dots$

9.37 / 9.40 Definitions (opposite / equivalent w.r.t. a subspace)
$a \mid_A^n b :\iff \mathrm{Sub}_n A \wedge a \notin A \wedge b \notin A \wedge \exists t\, [\, t \in A \wedge \mathrm{B}\, atb\,]$
$a \approx_A^n b :\iff \exists c\, [\, a \mid_A^n c \wedge b \mid_A^n c\,]$

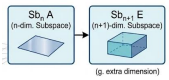
9.43 Definition (half-space) For $\mathrm{Sub}_n A$ and $a \notin A$: $\mathrm{Hs}_{A,a} := \{\, x \mid x \approx_A^n a \,\}$. As in the plane, $\approx_A^n$ divides the space outside of A into *exactly two disjoint half-spaces*.

9.44 Definition (subspace) The subspace of dimension $n+1$ defined by A ($\mathrm{Sub}_n A$) and $r \notin A$, with $r \mid_A^n r'$:
$$\mathrm{Sb}_{n+1}(A,r) := \mathrm{Hs}_{n+1}(A,r) \cup A \cup \mathrm{Hs}_{n+1}(A,r').$$
For $n = 1$ this is exactly Definition 9.20: half-spaces become half-planes and $\mathrm{Sb}_2(A,r) = \mathrm{Pl}(A,r)$.


$n = 2$: a, b opposite w.r.t. a plane.



Half-space $\mathrm{Hs}_{A,a}$ bounded by a plane.



$\mathrm{Sb}_n A \to \mathrm{Sb}_{n+1} E$: one extra dimension.

Recap: the same pattern in every dimension

boundary	relations	two sides (equivalence classes)	union = next dimension
line A (dim 1)	$a \mid_A b,\ a \simeq_A b$	half-planes $\mathrm{Hp}(A,r), \mathrm{Hp}(A,r')$	plane $\mathrm{Pl}(A,r)$ (dim 2)
$\mathrm{Sub}_n A$ (dim n)	$a \mid_A^n b,\ a \approx_A^n b$	half-spaces $\mathrm{Hs}(A,r), \mathrm{Hs}(A,r')$	$\mathrm{Sb}_{n+1}(A,r)$ (dim $n+1$)