Half-lines, Lines and Point Reflections

Handout · Seminar talk on Schwabhäuser/Szmielew/Tarski, *Metamathematische Methoden in der Geometrie*, Part I, Chapters 6 and 7 · Sophia Scherer

1 Basics

- Only **points** and two basic relations: **betweenness** B abc (b lies between a and c) and **congruence** ab ≡ cd (segment ab is congruent to segment cd).
- B is not strict: endpoints count, so B aab and B abb always hold. That is why many definitions demand a ≠ p.
- **Collinear** (4.10): Col pqx :⇔ B pqx ∨ B pxq ∨ B xpq.
- Chapters 6 and 7 use only **A1–A8** (absolute geometry, no parallel axiom A10). All results hold in Euclidean and in hyperbolic geometry.

Working method of the book

Syntax → Interpretation → Model

We derive theorems from the axioms (syntax). A model alone proves nothing; it shows what does *not* follow from the axioms.

A4 Segment construction

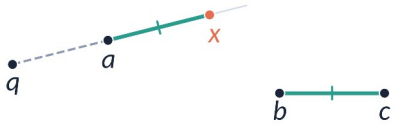


Fig. 1

∃x (B qax ∧ ax ≡ bc): extend qa beyond a by a segment congruent to bc.

A7 Pasch (inner form)

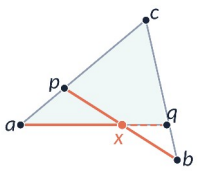


Fig. 2

B apc ∧ B bqc → ∃x (B pxb ∧ B qxa): intersection points only between existing points.

2 Chapter 6: Half-lines and lines

Def. 6.1 Same side. $a \equiv_p b :\Leftrightarrow a \neq p \wedge b \neq p \wedge (B\ pab \vee B\ pba)$.

a and b lie on **opposite sides** of p if p, a, b are pairwise distinct and B apb. ($\equiv_p$ has nothing to do with segment congruence.)



Fig. 4: same side

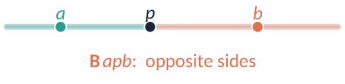


Fig. 3: opposite sides

Thm. 6.2 Opposite-point test.
a, b, c ≠ p ∧ B apc → (B bpc ⇔ $a \equiv_p b$).

With one fixed point c opposite a we can decide for every b on which side of p it lies.



If B*apc*: *b* on the same side as *a* ⟺ B*bpc*

Fig. 5: Theorem 6.2: c is the test point

Thms. 6.5–6.7. Let a, b, c ≠ p, and let a, b, c be collinear with p. Then $\equiv_p$ is an equivalence relation:

- reflexive: $a \equiv_p a$ (6.5)
- symmetric: $a \equiv_p b \rightarrow b \equiv_p a$ (6.6)
- transitive: $a \equiv_p b \wedge b \equiv_p c \rightarrow a \equiv_p c$ (6.7)

Two classes. Fix a ≠ p. Every x ≠ p with Col apx satisfies exactly one of: $x \equiv_p a$ (same side as a) or B xpa (opposite side) (6.4). So the points ≠ p that are collinear with a and p fall into exactly two equivalence classes; p belongs to neither.

Def. 6.8 Half-line. $H(pa) := \{ x \mid x \equiv_p a \}$.



Fig. 6: the two classes of $\equiv_p$ among the points collinear with a and p; a, b, c lie in one class, d in the other

Def. 6.14 Line. For p ≠ q: L(pq) := { x | Col pqx }.

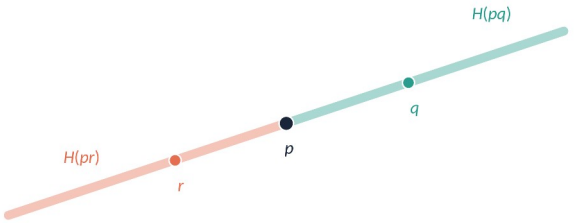


Fig. 7: a line: two opposite half-lines and p

Thm. 6.15. B qpr → L(pq) = H(pq) ∪ {p} ∪ H(pr).

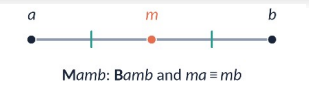
Thm. 6.18 and incidence theorems. If a ≠ b lie on a line A, then A = L(ab) (6.18).

Thm. 6.19 through two distinct points there is exactly one line
Thm. 6.21 two distinct lines share at most one point
Thm. 6.25 for every line there is a point outside it. In Hilbert's system these are axioms; here they are theorems.

Remark 6.26. Lines are not needed: every statement about lines can be written with points and Col alone.

3 Chapter 7: Point reflections

Def. 7.1 Midpoint. M amb :⇔ B amb ∧ ma ≡ mb.

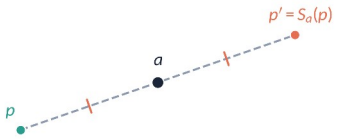


Mamb: Bamb and ma ≡ mb

Fig. 8: midpoint m of ab

Thm. 7.4 Mirror image. For all a, p there is exactly one p′ with M pap′ (existence by A4, uniqueness by 2.12).

Def. 7.5 Point reflection. We write p′ = S_a(p) (mirror image of p in a); S_a is the point reflection in a.



M pap′ : ⇔ B pap′ ∧ ap ≡ ap′

Fig. 9: mirror image p′ of p in a

Properties. 7.7: S_aS_ap = p. 7.10: S_ap = p ⇔ p = a. 7.11: S_a is bijective. 7.12: S_a is an involution.

Main theorem 7.13. pq ≡ S_a(p) S_a(q) – every segment is congruent to its image under a point reflection.

Corollary 7.14. S_a is a motion: it preserves B and ≡, and therefore also midpoints. Without angles and rotations, the proof uses only segments.

Tools: the five-segment axiom A5 (outer form) and Theorem 4.2 (inner form). Each compares figure 1 (a, b, c, d) with figure 2 (a′, b′, c′, d′); in both, b lies between a and c. Equal colours / tick marks = congruent segments, dashed = what follows.

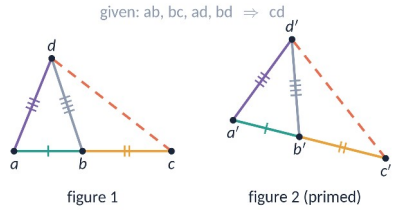


Fig. 10: Axiom A5 (OFS): ab, bc, ad, bd equal and a ≠ b ⇒ outer segment cd ≡ c′d′

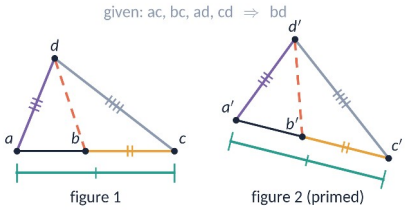


Fig. 11: IFS + Thm. 4.2: ac, bc, ad, cd equal ⇒ inner segment bd ≡ b′d′

Proof of 7.13 (p ≠ a; p′ = S_ap, q′ = S_aq):

Step 1 Frame (A4). Choose x, y, x′, y′ with B p′px, px ≡ qa; B q′qy, qy ≡ pa; B xp′x′, p′x′ ≡ qa; B yq′y′, q′y′ ≡ pa. By 2.11: ax ≡ ay ≡ ay′ ≡ ax′.

Step 2 Outer sides (A5). OFS(x a x′ y′ / y′ a y x) ⇒ x′y′ ≡ xy.

Step 3 Inner segment (4.2). IFS(y q a x / y′ q′ a x′) ⇒ qx ≡ q′x′.

Step 4 Inner segment (4.2). IFS(x p a q / x′ p′ a q′) ⇒ pq ≡ p′q′. ∎

Remark 7.24 Frame method. So far Pasch (A7) is the only source of intersection points, and it gives them only *between* existing points. So we first build a frame into which the sought point must fall.

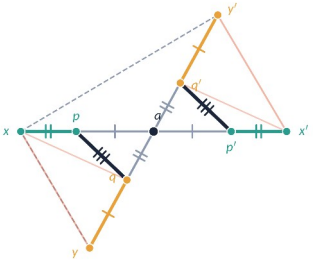


Fig. 12: the frame x, y, x′, y′ (tick marks: | ≡ ap, || ≡ aq)

4 Consequences and outlook

Thm. 7.17 Uniqueness of the midpoint. M pap′ ∧ M pbp′ → a = b. Idea: by 7.13, S_a(b) has the same distances to p and p′ as b, only swapped; by 4.19, S_a(b) = b, so b = a. (Existence of midpoints: Thm. 8.22.)

Thm. 7.19 Composition. S_aS_b p = S_bS_a p ⇔ a = b. The composition of point reflections is not commutative.

Outlook. 7.18: S_ap = S_b p → a = b. 7.20: Col amb ∧ ma ≡ mb → a = b ∨ M amb. 7.21: in a symmetric quadrilateral the diagonals bisect each other. 7.25: ca ≡ cb ⇒ ab has a midpoint.

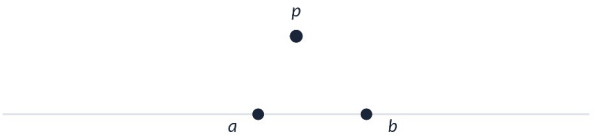


Fig. 13: draw it yourself: S_aS_bp (first b, then a) and S_bS_ap (first a, then b).

In short

- **Half-line** = equivalence class of $\equiv_p$; the key tool is the opposite-point test 6.2.
- **Line** = two opposite half-lines + their initial point; Hilbert's incidence axioms become theorems.
- **Point reflection** = motion (7.13, 7.14), proved with a frame and the five-segment statements. Consequences: the midpoint is unique (7.17), and $S_aS_b \neq S_bS_a$ for a ≠ b (7.19).

Sources

Schwabhäuser, W.; Szmielew, W.; Tarski, A. (1983): *Metamathematische Methoden in der Geometrie.* Hochschultext. Berlin/Heidelberg: Springer. (Part I, Chapters 6 and 7: definitions, theorems and numbering used throughout.)

Kasten, H.; Vogel, D. (2018): *Grundlagen der ebenen Geometrie.* Berlin: Springer Spektrum.

Kasten, H.; Vogel, D. (2025): *Einführung in die Geometrie.* Lecture notes (in German), summer term 2025, Heidelberg University.

All figures (Fig. 1–13) are own drawings based on the definitions and theorems of Schwabhäuser/Szmielew/Tarski (1983).