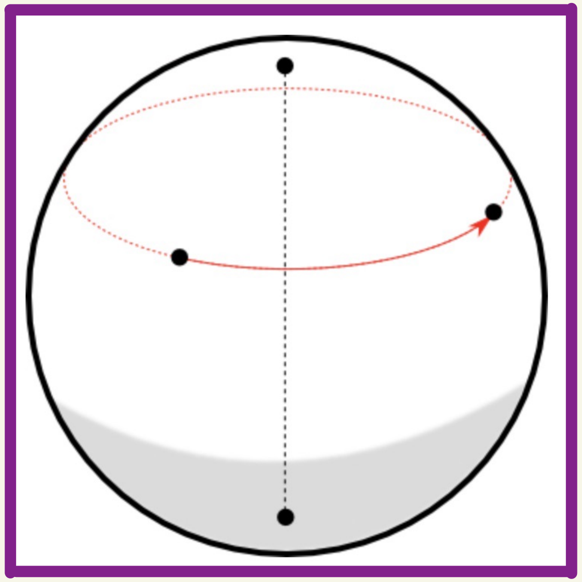
Group actions



HEGL seminar: Illustrating Mathematics
(WiSe 2023 - 2024)

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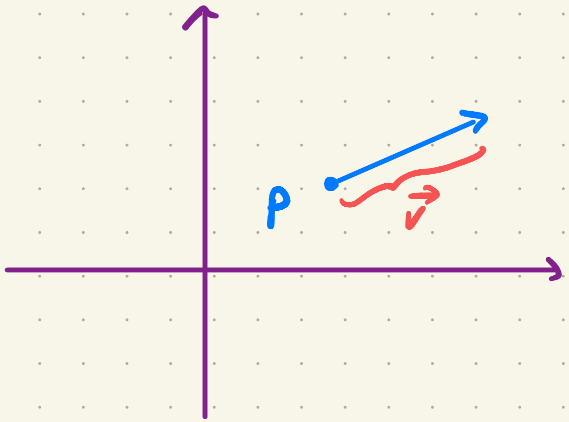
1. A basic example: translations in the plane

Let $P = \begin{pmatrix} x \\ y \end{pmatrix}$ be a point.

Let $\vec{v} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ be a vector.



Define a new point

$$\vec{v} + P := \begin{pmatrix} \alpha + x \\ \beta + y \end{pmatrix}$$

Then $\vec{0} + P = P$ and $\vec{w} + (\vec{v} + P) = (\vec{w} + \vec{v}) + P$.

$\rightsquigarrow$ we say that the group of vectors acts on points in the plane by translation

2. Formal definition and examples

Definition Let G be a group. An action of G on X is a map

$$\begin{aligned} G \times X &\xrightarrow{a} X \\ (g, x) &\longmapsto g \cdot x = a(g, x) \end{aligned}$$

notation

such that:

(i) $\forall x \in X, \quad e \cdot x = x$

neutral element in G

(ii) $\forall g_1, g_2 \in G, \ \forall x \in X,$

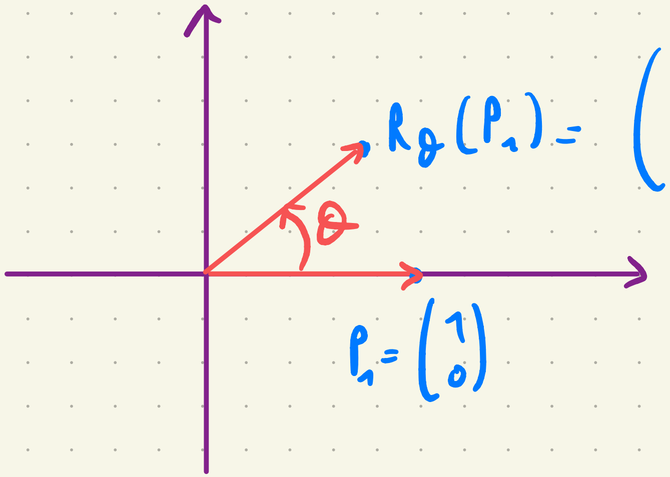
$$g_1 \cdot (g_2 \cdot x) = \underbrace{(g_1 g_2)}_{\text{"product" in } G} \cdot x$$

Example 1 Rotations in the plane

$$P = \begin{pmatrix} x \\ y \end{pmatrix} \qquad R_\theta = \begin{pmatrix} \cos\theta & -\operatorname{sen}\theta \\ \operatorname{sen}\theta & \cos\theta \end{pmatrix}$$

(θ a real number)

$$R_\theta(P) = \begin{pmatrix} \cos\theta & -\operatorname{sen}\theta \\ \operatorname{sen}\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (\cos\theta)x - (\operatorname{sen}\theta)y \\ (\operatorname{sen}\theta)x + (\cos\theta)y \end{pmatrix}$$



$R_\theta(P_1) = \begin{pmatrix} \cos\theta \\ \operatorname{sen}\theta \end{pmatrix}$

$P_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Exercise Show that

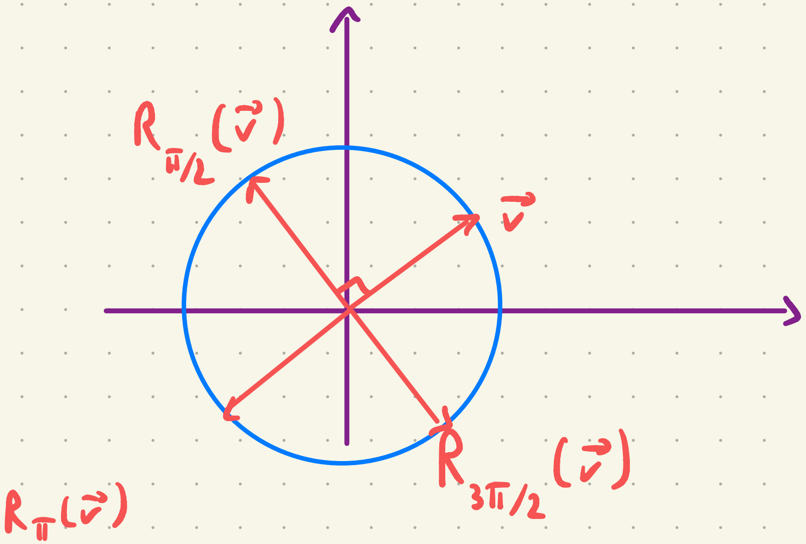
$$\underbrace{R_\theta \circ R_{\theta'} = R_{\theta+\theta'}}_{}.$$

such rotations R_θ form a group denoted $SO(2)$.

It acts on the Euclidean plane $\mathbb{R}^2$.

Example 2

group generated by $R_{\pi/2}$



$$R_{\pi/2} \circ R_{\pi/2} \circ R_{\pi/2} \circ R_{\pi/2} = Id$$

$$\langle R_{\pi/2} \rangle = \underbrace{\{ I.d ; R_{\pi/2} ; R_{\pi} ; R_{3\pi/2} \}}_{\text{subgroup of the group of rotations}}$$

+ relation

$$R_{\pi/2}^{4} = Id$$

Example 3

$$GL(2;\mathbb{R}) \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\left(\begin{pmatrix} a & c \\ b & d \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right) \longmapsto \begin{pmatrix} ax + cy \\ bx + dy \end{pmatrix}$$

$$\mathbb{R}^* \times \mathbb{R} \longrightarrow \mathbb{R}$$

$$(\lambda , x) \longmapsto \lambda x$$

Exercise To give a group action

$$a : G \times X \to X$$

is equivalent to giving a group homomorphism

$$\Phi : G \to \mathrm{Bij}(X) \quad \} \text{ bijective maps } X \to X$$

Hint: Define $\Phi(g) := a(g, \cdot)$.

3. Orbits, stabilizers, orbit spaces

Recall the action map $\begin{aligned} a : G \times X &\longrightarrow X . \\ (g, x) &\longmapsto g \cdot x \end{aligned}$

Fix a point $x \in X$. This defines a map

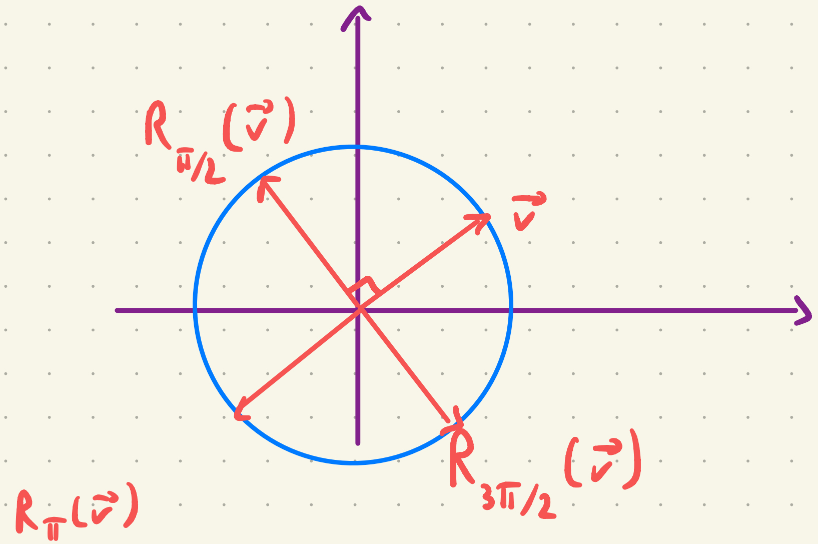
$$\begin{aligned} G &\xrightarrow{a_x} X \\ g &\longmapsto g \cdot x \end{aligned}$$

Definition The image of the map a_x is called the orbit of x, and is denoted $G \cdot x$.

$$G \cdot x = \{ x' \in X \mid \exists g \in G, x' = g \cdot x \}$$

Example

Here, the orbit of $\vec{v}$ contains four vectors.



Exercise If $(G \cdot x) \cap (G \cdot x') \neq \emptyset$,

then $G \cdot x = G \cdot x'$.

[two orbits are either disjoint or equal]

Definition

If the action of G on X has only one orbit, then it is called a transitive action.

i.e. $\forall x, x' \in X, \; \exists g \in G, \; x' = g \cdot x$.

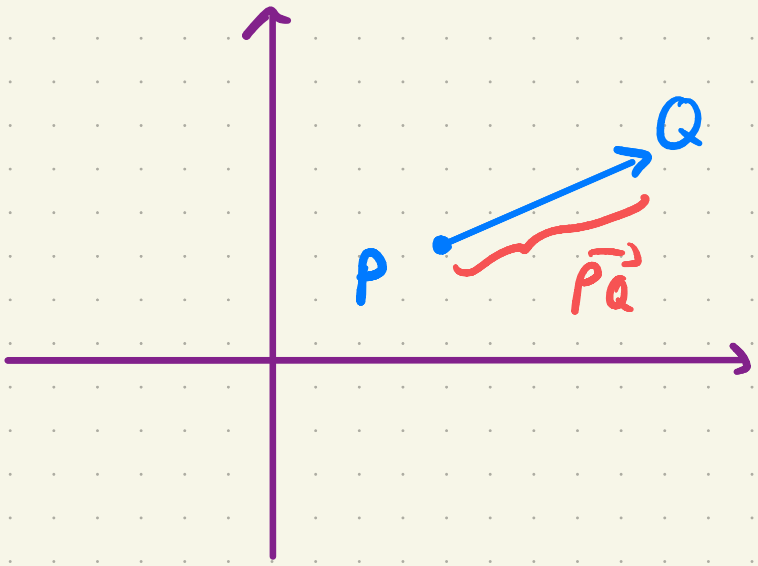
$$(G \cdot x = X)$$

Example

given P and Q,

there exists $\vec{v}$ such that $\vec{v} + P = Q$.

---> take $v = \overrightarrow{PQ}$.



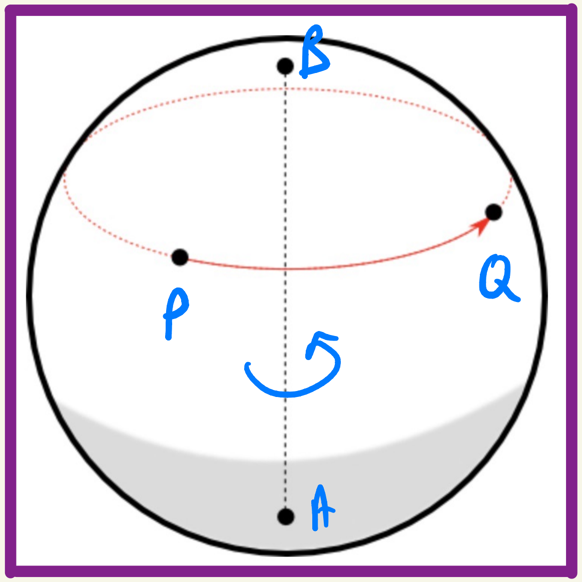
Definition given $x \in X$, the set

$$G_x := \{ g \in G \mid g \cdot x = x \}$$

is called the stabilizer of x in G.

Example

The rotation taking P to Q pictured here belongs to the stabilizer of A (and of B).



Proposition The stabilizer G_x is a subgroup of G.

Proof:

- $e \in G_x$ because $e \cdot x = x$ by the first axiom in the definition of a group action.

- if $g_1, g_2 \in G_x$, then $g_1 g_2 \in G_x$ because

$$\begin{aligned} (g_1 g_2) \cdot x &= g_1 \cdot (g_2 \cdot x) \\ &= g_1 \cdot x \\ &= x \end{aligned}$$

second axiom

$g_2 \in G_x$

$g_1 \in G_x$

- if $g \in G_x$, then $g^{-1} \in G_x$, because $g \cdot x = x$ so

$$\begin{aligned} g^{-1} \cdot x &= g^{-1} \cdot (g \cdot x) \\ &= (g^{-1} g) \cdot x \\ &= e \cdot x \\ &= x . \end{aligned}$$

$\square$

Exercise Two points in the same orbit have conjugate stabilizers:

$$G_{g \cdot x} = g \, G_x \, g^{-1}$$

(equivalently, $h \in G_x$ iff $g h g^{-1} \in G_{g \cdot x}$).

Theorem

"Orbit-stabilizer theorem"

For all $x \in X$, the map

$$\begin{aligned} G &\xrightarrow{a_x} X \\ g &\longmapsto g\cdot x \end{aligned}$$

induces a bijection

$$\underbrace{G/G_x}_{\text{set of } \underbrace{\text{right } G_x\text{-cosets}}_{gG_x,\ g \in G} \text{ in } G} \cong G\cdot x$$

✓

i.e. $\operatorname{Im} a_x = G\cdot x$

and $\underbrace{a_x(g) = a_x(g') \text{ iff } g' \in gG_x}_{g\cdot x = g'\cdot x \text{ iff } (g^{-1}g')\cdot x = x}$

i.e. $g^{-1}g' \in G_x$.

Corollary 1 Let G be a finite group.

Then

$$\left.\begin{aligned} \operatorname{card}(G\cdot x) &= \operatorname{card}(G/G_x) \\ &\left(= [G : G_x]\right) \\ &= \frac{\operatorname{card}(G)}{\operatorname{card}(G_x)} . \end{aligned}\right\}$$

because G is the disjoint union of the right cosets $\underbrace{g G_x}$

same cardinal as G_x, because

$$G_x \to g G_x$$

$$h \mapsto g h$$

is a bijection.

In particular, the cardinal of an orbit divides the cardinal of the group:

$$\operatorname{card}(G\cdot x)\, \operatorname{card}(G_x) = \operatorname{card}(G)$$

"The cardinal of an orbit is equal to the index of the stabilizer of any of its points"

Exercise The relation $x \sim x'$ if $x' \in G \cdot x$ is an equivalence relation on X. (whose equivalence classes are precisely the orbits of G in X).

Definition The set of equivalence classes of the equivalence relation $x \sim x'$ if x and x' lie in the same orbit is called the orbit space of the action, and is denoted X/G.

The elements of X/G are the orbits of G in X.

Theorem (Class formula)

Let G be a finite group, acting on a finite set X. Then

$$\mathrm{card}(X) = \mathrm{card}(G) \sum_{G.x \in X/G} \frac{1}{\mathrm{card}(G_x)}$$

Proof The orbit space X/G defines a partition of X. So $\mathrm{card}(X) = \sum_{G.x \in X/G} \mathrm{card}(G.x)$.

By the orbit-stabilizer theorem, $\mathrm{card}(G.x) = \dfrac{\mathrm{card}(G)}{\mathrm{card}(G.x)}$, hence the conclusion.

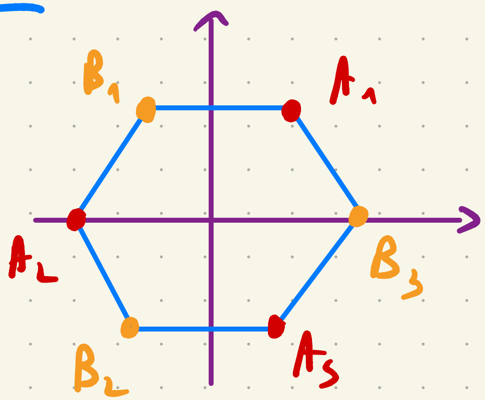
□

Definition If $\forall x \in X$, $G_x = \{e\}$, then the action of G on X is called a free action.

Example

- Translations in the plane.
- Rotations of angle $\frac{2\pi}{n}$ acting on the vertices of a regular polygon with $2n$ sides.

$n=3$



$6 = 3 \times 2$

free action with 2 orbits; each orbit has n points.

$$\mathrm{card}(X) = \mathrm{card}(G) \sum_{G.x \in X/G} \frac{1}{1}$$

$$= \mathrm{card}(G)\, \mathrm{card}(X/G).$$

Observation For finite G and X, the number of orbits of a free action is

$$\mathrm{card}\,(X/G) = \frac{\mathrm{card}\,(X)}{\mathrm{card}\,(G)}.$$

Another way of counting orbits, without assuming that the action is free, is via the Burnside formula.

Definition Take $g \in G$ and set

$$\mathrm{Fix}_g\,(X) = \{ x \in X \mid g \cdot x = x \}$$

(the set of elements in X that are fixed by g).

Theorem (Burnside formula)

For finite G and X,

$$\mathrm{card}\,(X/G) = \frac{1}{\mathrm{card}\,(G)} \sum_{g \in G} \mathrm{card}\ \mathrm{Fix}_g(X)$$

Proof The key is that

$$\sum_{x \in X} \mathrm{card}(G_x) \overset{(*)}{=} \sum_{g \in G} \mathrm{card}\ \mathrm{Fix}_g(X)$$

(by counting the number of elements in

$$\{ (g, x) \in G \times X \mid g \cdot x = x \}$$

in two different ways).

Then we use $\mathrm{card}\,(G_x) = \dfrac{\mathrm{card}\,(G)}{\mathrm{card}\,(G \cdot x)}$ and we partition X into (finitely many) orbits.

Explicitly,

$$\underbrace{\sum_{x \in X} \text{card}(G_x)}_{= \sum_{g \in G} \text{card}\, \text{Fix}_g(X)} = \text{card}(G) \sum_{G.x \in X/G} \sum_{y \in G.x} \frac{1}{\text{card}(G.x)}$$

$$= \text{card}(G)\, \text{card}(X/G).$$

$\square$

Exercise If $g' = hgh^{-1}$, then there is a well-defined map

$$\text{Fix}_g(X) \longrightarrow \text{Fix}_{g'}(X)$$
$$x \longmapsto h.x$$

and this map is a bijection.

Application

If a (finite) group G acts on itself by conjugation

$$\begin{aligned} G \times G &\xrightarrow{a} G \\ (g, h) &\mapsto ghg^{-1} \end{aligned}$$

the fixed-point set $\mathrm{Fix}_g(G)$ is called the centralizer of g in G:

$$\mathcal{C}_G(g) = \{ h \in G \mid hg = gh \}.$$

The orbits of that action are called conjugacy classes, and the Burnside formula says that the number of such conjugacy classes is equal to

$$\frac{1}{\mathrm{card}(G)} \sum_{g \in G} \mathcal{C}_G(g).$$

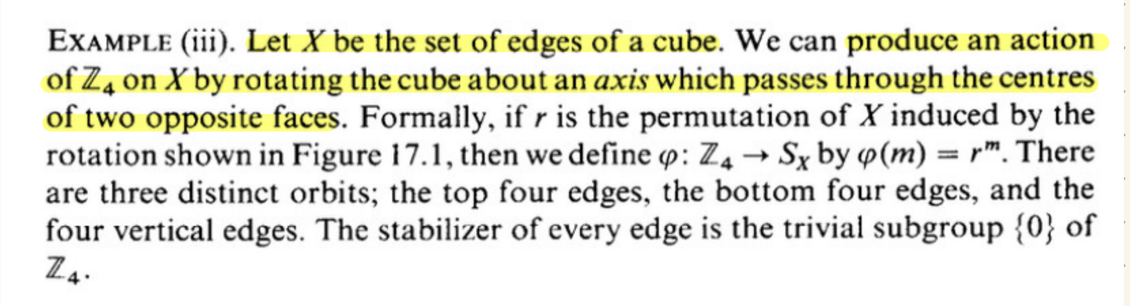
Exercise (for later)

Show that the number of conjugacy classes in the symmetric group S_3 (permutations of the set $\{1; 2; 3\}$) is equal to 3.

This involves computing the centralizer of each of the six elements of S_3, which is generated by the transposition $\sigma = (1\ 2)$ and the 3-cycle $\tau = (1\ 2\ 3)$. Explicitly,

$$S_3 = \{ id, \sigma, \tau, \sigma\tau, \tau^2, \sigma\tau^2 \}$$

and the three conjugacy classes are that of id, σ and τ. They contain respectively, 1, 3 and 2 elements.



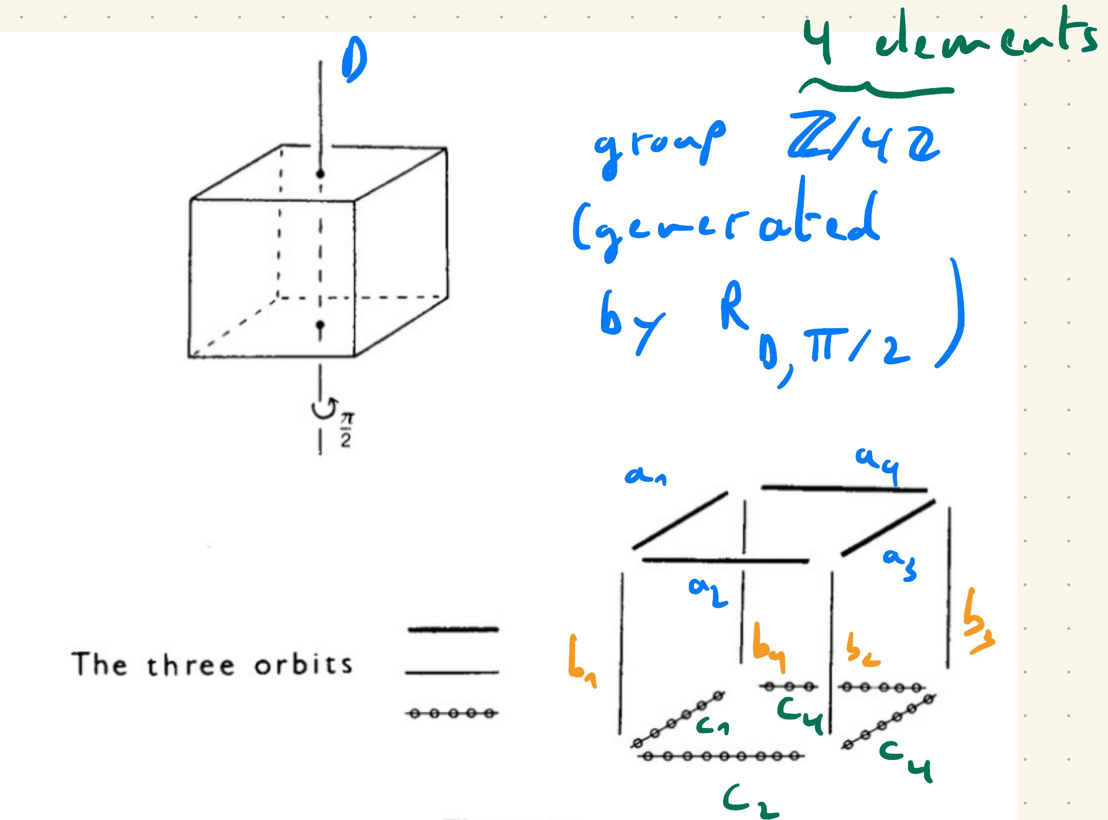
EXAMPLE (iii). Let X be the set of edges of a cube. We can produce an action of $\mathbb{Z}_4$ on X by rotating the cube about an *axis* which passes through the centres of two opposite faces. Formally, if r is the permutation of X induced by the rotation shown in Figure 17.1, then we define $\varphi: \mathbb{Z}_4 \to S_X$ by $\varphi(m) = r^m$. There are three distinct orbits; the top four edges, the bottom four edges, and the four vertical edges. The stabilizer of every edge is the trivial subgroup $\{0\}$ of $\mathbb{Z}_4$.

Pages 92-93 of Armstrong's book

Free action (why?)

Three orbits, each one with four elements.

X has $\underbrace{4}_{\text{card } \mathbb{Z}/4\mathbb{Z}} \times \underbrace{4}_{\text{card orbit}} = 16$

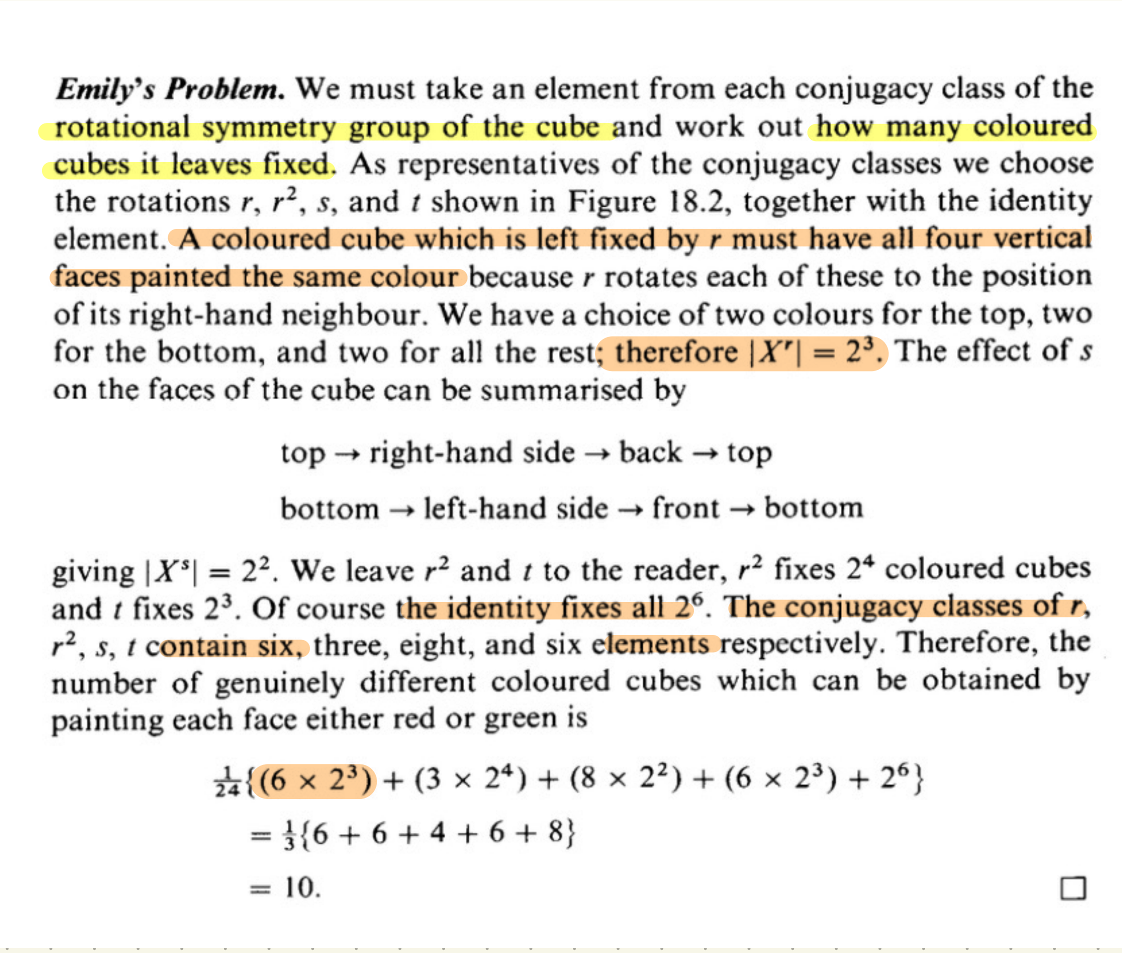


group $\underbrace{\mathbb{Z}/4\mathbb{Z}}^{4 \text{ elements}}$ (generated by $R_{D,\pi/2}$)

Figure 17.1

An application of Burnside Formula

To count the number of coloured cubes, count first the number of coloured cubes that are fixed by g, for all $g \in S_4$.



Emily's Problem. We must take an element from each conjugacy class of the rotational symmetry group of the cube and work out how many coloured cubes it leaves fixed. As representatives of the conjugacy classes we choose the rotations r, r^2, s, and t shown in Figure 18.2, together with the identity element. A coloured cube which is left fixed by r must have all four vertical faces painted the same colour because r rotates each of these to the position of its right-hand neighbour. We have a choice of two colours for the top, two for the bottom, and two for all the rest; therefore $|X^r| = 2^3$. The effect of s on the faces of the cube can be summarised by

top → right-hand side → back → top

bottom → left-hand side → front → bottom

giving $|X^s| = 2^2$. We leave r^2 and t to the reader, r^2 fixes 2^4 coloured cubes and t fixes 2^3. Of course the identity fixes all 2^6. The conjugacy classes of r, r^2, s, t contain six, three, eight, and six elements respectively. Therefore, the number of genuinely different coloured cubes which can be obtained by painting each face either red or green is

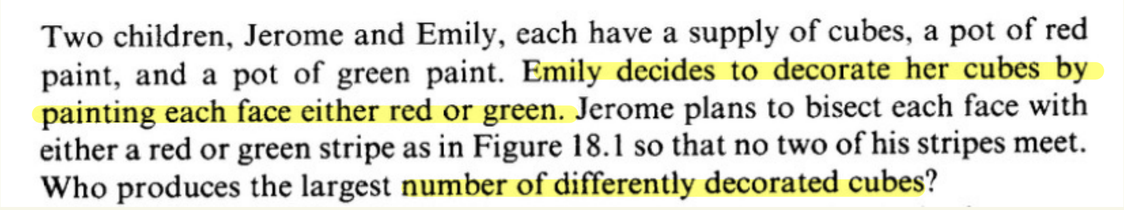
$$\begin{aligned}&\tfrac{1}{24}\{(6 \times 2^3) + (3 \times 2^4) + (8 \times 2^2) + (6 \times 2^3) + 2^6\}\\ &= \tfrac{1}{3}\{6 + 6 + 4 + 6 + 8\}\\ &= 10.\end{aligned}$$

□

Note that it suffices to do that for <u>one</u> g in each conjugacy class, and to multiply the result by the number of elements in the conjugacy class.

Example Symmetries of the cube

[Armstrong's book pp. 98 – 101]



Two children, Jerome and Emily, each have a supply of cubes, a pot of red paint, and a pot of green paint. Emily decides to decorate her cubes by painting each face either red or green. Jerome plans to bisect each face with either a red or green stripe as in Figure 18.1 so that no two of his stripes meet. Who produces the largest number of differently decorated cubes?

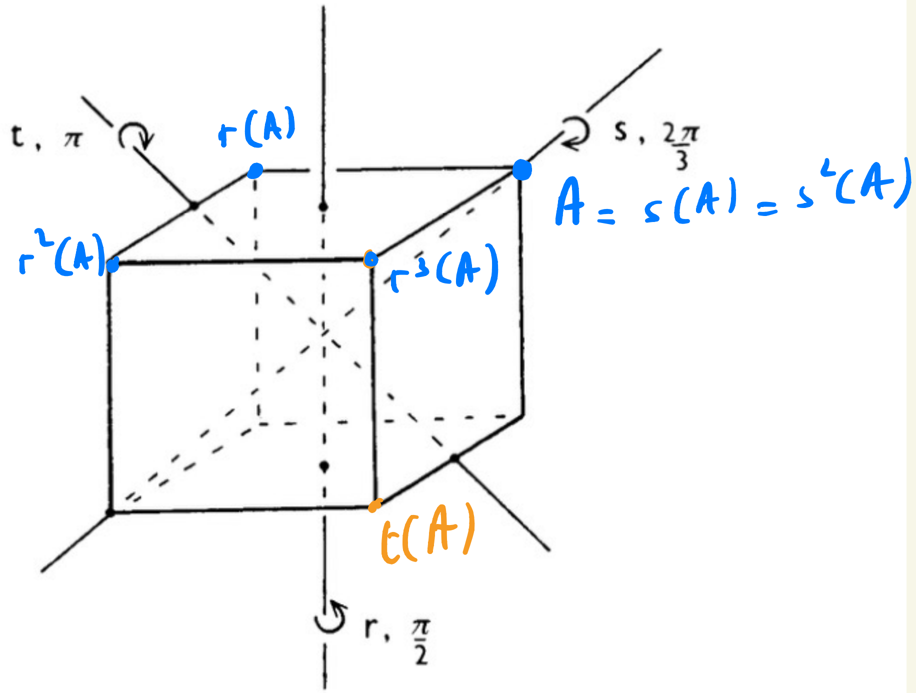


Figure 18.2

The group of rotations in $\mathbb{R}^3$ that preserve the unit cube is isomorphic to S_4, which has order 24.

4. Semidirect products

The group of direct rotations of the plane | $\mathbb{R}^2$ | $\mathbb{E}^2$

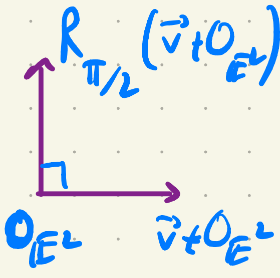
$$R_\theta = \begin{pmatrix} \cos\theta & -\mathrm{sen}\,\theta \\ \mathrm{sen}\,\theta & \cos\theta \end{pmatrix} \qquad (\theta \text{ a real number})$$

is denoted by $SO(2)$.

The group of translations can be denoted by $\mathbb{R}^2$.

Point $SO(2)$ and $\mathbb{R}^2$ both act on $\mathbb{E}^2$. But, the actions do not commute!

⇝ In general, $R_\theta(\vec{v} + P) \neq \vec{v} + R_\theta(P)$.



Example Take $P = 0_{\mathbb{E}^2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
Then $R_{\pi/2}(0_{\mathbb{E}^2}) = 0_{\mathbb{E}^2}$ but $R_{\pi/2}(\vec{v} + 0_{\mathbb{E}^2}) \neq \vec{v} + 0_{\mathbb{E}^2}$.

Proposition $R_\theta(\vec{v} + P) = R_\theta(\vec{v}) + R_\theta(P)$

Proof Exercise!

(the issue is to understand the meaning of $R_\theta(\vec{v})$).

$\square$

Then we can construct a group denoted $\mathbb{R}^2 \rtimes SO(2)$, whose elements are pairs $(\vec{v}, R_\theta)$ with $\vec{v} \in \mathbb{R}^2$, $R_\theta \in SO(2)$ and product defined by

$$\underbrace{(\vec{v}, R_\theta)(\vec{w}, R_\alpha) = (\vec{v} + R_\theta(\vec{w}), R_{\theta+\alpha})}_{\text{check that this defines a group!}}.$$

The point is that the group $\mathbb{R}^2 \rtimes SO(2)$ just defined contains $\mathbb{R}^2$ and $SO(2)$ as subgroups (with $\mathbb{R}^2$ a normal subgroup) and, more importantly for us, that this group acts on $\mathbb{E}^2$ via

$$\underbrace{(\vec{v}, R_\theta) \cdot P = \vec{v} + R_\theta(P).}_{\text{check that this defines an action of } \mathbb{R}^2 \rtimes SO(2) \text{ on } \mathbb{E}^2.}$$

Exercise Prove that the following maps are group homomorphisms

$$\begin{array}{ccccc} \mathbb{R}^2 & \xrightarrow{i} & \mathbb{R}^2 \rtimes SO(2) & \xrightarrow{\pi} & SO(2) \\ \vec{v} & \longmapsto & (\vec{v}, \mathrm{Id}_{\mathbb{E}^2}) & & \\ & & (\vec{v}, R_\theta) & \longmapsto & R_\theta \end{array}$$

and that

1. i is injective
2. π is surjective
3. $\operatorname{Im} i = \operatorname{Ker} \pi$
4. The map
$$s : SO(2) \longrightarrow \mathbb{R}^2 \rtimes SO(2)$$
$$R_\theta \longmapsto (0_{\mathbb{R}^2}, R_\theta)$$
is a group homomorphism satisfying
$$\pi \circ s = Id_{SO(2)} \, .$$