Theorem: Let $(K, \mathcal{T})$ be a topological space. Let $S \subseteq K$. Then there are at most 14 pairwise distinct sets that can be formed by repeatedly applying the operations "closure" and "complement" to S.

Proof: Let kS denote the closure of S, and cS the complement of S. It suffices to prove the following three identities:

1. $kkS = kS$ (the closure operation is idempotent)
2. $ccS = S$ (the complement operation is an involution)
3. $kckckckcS = kckcS$ (the operation $kckc$ is an involution)

The 14 distinct sets that arise from these operations are:

S	cS
kS	ckS
kcS	$ckcS$
$kckS$	$ckckS$
$kckcS$	$ckckcS$
$kckckS$	$ckckckS$
$kckckcS$	$ckckckcS$

To 1): kS is the intersection of all closed subsets X of K that contain S. Since kS is itself closed, it follows that $kkS = kS$.

To 2): $x \in S \Leftrightarrow x \notin cS \Leftrightarrow x \in ccS$.

To 3): We first prove the following auxiliary statements:

i) $A \subseteq B \Rightarrow iA \subseteq iB$
Proof: Let $x \in iA$. Then there exists an open set $E \subseteq A$ with $x \in E$. Since $E \subseteq A \subseteq B$, we have $x \in iB$.

ii) $A \subseteq B \Rightarrow kA \subseteq kB$
Proof: Let $x \in kA$, and let M be a closed set with $B \subseteq M$. Then $A \subseteq B \subseteq M \Rightarrow kA \subseteq M \Rightarrow x \in M \Rightarrow x \in kB$.

iii) $iS = ckcS$
Proof:

- $iS \subseteq ckcS$: Let $x \in iS$. Then there exists an open set $E \subseteq S$ with $x \in E$. Since E is open, cE is closed and $cS \subseteq cE \Rightarrow kcS \subseteq cE \Rightarrow x \notin kcS \Rightarrow x \in ckcS$.
- $ckcS \subseteq iS$: Let $x \in ckcS$. Since kcS is closed, $ckcS$ is open, and there exists an open set $E \subseteq ckcS$ with $x \in E$ and $E \cap kcS = \emptyset$. Then $E \cap cS = \emptyset \Rightarrow E \subseteq S$, and since E is open, $x \in iS$.

We now show that $kikiS = kiS$. From (iii) it then follows that $kckckckcS = kckcS$.
Proof:

- $kikiS \subseteq kiS$: $ikiS \subseteq kiS \Rightarrow kikiS \subseteq kkiS = kiS$ (i), (ii)
- $kiS \subseteq kikiS$: $iS \subseteq kiS \Rightarrow iiS \subseteq ikiS \Rightarrow kiS \subseteq kikiS$ (i), (ii)

The maximum of 14 pairwise distinct sets can be realised in $\mathbb{R}$: Let $\mathrm{K} = \mathbb{R}$, $\mathrm{S} = (0,1) \cup (1,2) \cup \{3\} \cup ([4,5] \cap \mathbb{Q})$. The 14 sets are the following:

1. $\mathrm{S} = (0,1) \cup (1,2) \cup \{3\} \cup ([4,5] \cap \mathbb{Q})$
2. $\mathrm{cS} = (-\infty,0] \cup \{1\} \cup [2,3) \cup (3,4) \cup ((4,5) \setminus \mathbb{Q}) \cup (5,\infty)$
3. $\mathrm{kcS} = (-\infty,0] \cup \{1\} \cup [2,\infty)$
4. $\mathrm{ckcS} = (0,1) \cup (1,2)$
5. $\mathrm{kckcS} = [0,2]$
6. $\mathrm{ckckcS} = (-\infty,0) \cup (2,\infty)$
7. $\mathrm{kckckcS} = (-\infty,0] \cup [2,\infty)$
8. $\mathrm{ckckckcS} = (0,2)$
9. $\mathrm{kS} = [0,2] \cup \{3\} \cup [4,5]$
10. $\mathrm{ckS} = (-\infty,0) \cup (2,3) \cup (3,4) \cup (5,\infty)$
11. $\mathrm{kckS} = (-\infty,0] \cup [2,4] \cup [5,\infty)$
12. $\mathrm{ckckS} = (0,2) \cup (4,5)$
13. $\mathrm{kckckS} = [0,2] \cup [4,5]$
14. $\mathrm{ckckckS} = (-\infty,0) \cup (2,4) \cup (5,\infty)$